

## Bethe equations for generalized Hubbard models

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## Bethe equations for generalized Hubbard models

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**ABSTRACT:** We compute the eigenfunctions, energies and Bethe equations for a class of generalized integrable Hubbard models based on  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$  superalgebras. The Bethe equations appear to be similar to the Hubbard model ones, up to a phase due to the integration of a subset of ‘simple’ Bethe equations. We discuss relations with AdS/CFT correspondence, and with condensed matter physics.

**KEYWORDS:** Integrable Equations in Physics, Bethe Ansatz

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## 1 Introduction

The Hubbard model was first introduced as a model to describe the effects of the strongly correlated d-electrons in transition metals [1, 2]. It was shown that this model possesses a Mott metal-insulator phase transition [3, 4] and is relevant to the studies of high- $T_c$  superconductivity [5, 6]. Few exact results are known for the two or three-dimensional Hubbard model but nevertheless these models are still actively investigated.

In contrast, the one-dimensional Hubbard model is integrable and was first diagonalized by means of coordinate Bethe Ansatz by Lieb and Wu [7]. However its integrable

structure is rather complicated in comparison with the usual formalism of the spin chains (for a review see e.g. [8] and references therein). The Hubbard model  $R$ -matrix was first introduced by Shastry [9] and Olmedilla et al. [10] (by coupling two XX model  $R$ -matrices with U-interaction term). The proof of the Yang-Baxter relation was given by Shiroishi and Wadati [11]. A lot of extended Hamiltonians were also proposed to connect the model with high- $T_c$  superconductivity effects [12–14] (and references therein).

Generalization of one-dimensional Hubbard model to  $gl(\mathfrak{n})$  case via  $R$ -matrix was made by Maassarani et al. [15]. Extension of Maassarani’s approach to superalgebras  $gl(\mathfrak{n}|\mathfrak{m})$  was given in [16], and a “universal” presentation (including a generalization of Maassarani’s approach) was proposed in [17]. It is based on the decomposition of an arbitrary vector space into a direct sum of two subspaces, the two corresponding orthogonal projectors allowing one to define an  $R$ -matrix of a universal XX model, and then of a Hubbard model using a Shastry type construction. The construction is very general, since the two XX models that are used can be based on different (super)algebras. The QISM approach ensures the integrability of the models, leading to local Hubbard-like Hamiltonians thanks to the appropriate behavior of the obtained  $R$ -matrices.

Recent studies of the Hubbard model and its generalizations, were motivated by its “recent” applications in  $N = 4$  super Yang-Mills theory (SYM) — for example, see [18] and references therein. Although it appeared that the one-dimensional Hubbard model is not the proper answer due to transcendental contributions to the anomalous dimensions (in the  $su(2)$  subsector of the theory), one may find new directions in this field by studying integrable extensions of the one-dimensional Hubbard model. Another aspect lies in the possibility of applications to condensed matter physics, particularly when dealing with small rank algebras.

In this paper, we focus on a subcase of the “universal” presentation where the super-algebra for one XX model is  $gl(\mathfrak{n}|\mathfrak{m})$  while the other XX model is based on the algebra  $gl(2)$ , the projectors for both XX models being one-dimensional. This kind of model is simple enough to investigate further the problem of Bethe Ansatz Equations (see [17]), and presents interesting features to find some applications in condensed matter physics as well as some relevance in AdS/CFT correspondence.

The paper is organised as follows. In section 2 we prepare the notations and recall some results on generalized Hubbard model based on unitary (super)algebras. In section 3, we solve via the coordinate Bethe ansatz a toy model constructed on  $gl(2|1) \oplus gl(2)$  with one-dimensional projectors and we generalize the obtained results to  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$  models in section 4. In section 5, we study a model based on  $gl(2|2) \oplus gl(2)$ , where the projector for the  $gl(2|2)$  part is now two-dimensional. We argue on the relevance of this model for the AdS/CFT correspondence. In section 6, we perform the Jordan-Wigner transformation for  $gl(2|1) \oplus gl(2)$  and  $gl(2|2) \oplus gl(2)$  models to construct physical Hamiltonians.

## 2 Some results on $gl(\mathfrak{n}|\mathfrak{m})$ Hubbard model

We remind in this section the necessary notations for self-consistency and understanding of the paper. For more details, we refer the reader to [17].

The starting point is the definition of the  $R$ -matrix of an XX model based on the superalgebra  $\mathcal{G} = gl(\mathfrak{n}|\mathfrak{m})$ . Consider the graded vector space  $\mathcal{V} = \mathbb{C}^{\mathfrak{n}|\mathfrak{m}}$  (with possibly  $\mathfrak{m} = 0$ ) and define the  $\mathbb{Z}_2$ -grading on indices  $j$  by

$$[j] = \begin{cases} 0 & \text{for } 1 \leq j \leq \mathfrak{n} \\ 1 & \text{for } \mathfrak{n} + 1 \leq j \leq \mathfrak{n} + \mathfrak{m}. \end{cases} \quad (2.1)$$

Accordingly, the elementary matrices  $E^{ij}$  (with entry 1 at row  $i$  and column  $j$  and 0 elsewhere) have grade  $[E^{ij}] = [i] + [j]$ . We introduce a set of integers  $\mathcal{N} \subset \{1, \dots, \mathfrak{n} + \mathfrak{m}\}$  and  $\overline{\mathcal{N}}$  its complementary subset in  $\{1, \dots, \mathfrak{n} + \mathfrak{m}\}$ . We then define the following projectors:

$$\pi = \sum_{j \in \mathcal{N}} E^{jj}, \quad \overline{\pi} = \pi^{(\overline{\mathcal{N}})} = \sum_{\overline{j} \in \overline{\mathcal{N}}} E^{\overline{j}\overline{j}} \quad (2.2)$$

as well as (using auxiliary space notation)

$$\Sigma_{12} = \pi_1 \overline{\pi}_2 + \overline{\pi}_1 \pi_2. \quad (2.3)$$

The  $R$ -matrix of a XX model based on  $gl(\mathfrak{n}|\mathfrak{m})$  with projectors  $(\pi, \overline{\pi})$  is given by:

$$R_{12}(\lambda) = \Sigma_{12} P_{12} + \Sigma_{12} \sin \lambda + (\mathbb{I} \otimes \mathbb{I} - \Sigma_{12}) P_{12} \cos \lambda \quad (2.4)$$

where  $P_{12}$  is the graded permutation operator and  $\lambda \in \mathbb{C}$  the spectral parameter. It obeys Yang-Baxter equation, is unitary and regular.

The  $R$ -matrix for (generalized) Hubbard models is obtained by coupling the  $R$ -matrices  $R_{12}^\uparrow(\lambda)$  and  $R_{12}^\downarrow(\lambda)$  of two independent XX models, the coupling constant being related to the potential  $U$  of the Hubbard model under consideration [8, 9, 15–17]. We stress that the two XX models can be based on two different (super)algebras  $\mathcal{G}_\uparrow$  and  $\mathcal{G}_\downarrow$ , with two different (graded) vector spaces  $\mathcal{V}_\uparrow$  and  $\mathcal{V}_\downarrow$  and two different projectors  $\pi_\uparrow$  and  $\pi_\downarrow$ , associated to two different sets  $\mathcal{N}_\uparrow$  and  $\mathcal{N}_\downarrow$ .

Introducing the parity matrix  $C_\alpha$  ( $\alpha = \uparrow$  or  $\downarrow$ ):

$$C_\alpha = \sum_{j \in \mathcal{N}} E_\alpha^{jj} - \sum_{\overline{j} \in \overline{\mathcal{N}}} E_\alpha^{\overline{j}\overline{j}} = \pi_\alpha - \overline{\pi}_\alpha, \quad (2.5)$$

the  $R$ -matrix of the Hubbard model based on the pair of (super)algebras  $\mathcal{G}_\uparrow$  and  $\mathcal{G}_\downarrow$  is given by:

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) = R_{12}^\uparrow(\lambda_{12}) R_{12}^\downarrow(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})} \tanh(h'_{12}) R_{12}^\uparrow(\lambda'_{12}) C_{\uparrow 1} R_{12}^\downarrow(\lambda'_{12}) C_{\downarrow 1} \quad (2.6)$$

where  $\lambda_{12} = \lambda_1 - \lambda_2$  and  $\lambda'_{12} = \lambda_1 + \lambda_2$ . In the same way,  $h'_{12} = h(\lambda_1) + h(\lambda_2)$  and the function  $h(\lambda)$  is such that

$$\sinh(2h) = U \sin(2\lambda). \quad (2.7)$$

Note that the site 1 for  $R_{12}^{\uparrow\downarrow}$  is composed from the tensor product of the site “1  $\uparrow$ ” appearing in the matrix  $R_{12}^\uparrow$  by the site “1  $\downarrow$ ” which is in the matrix  $R_{12}^\downarrow$ . This is obviously the same for any site we will consider in the following.

The  $R$ -matrix (2.6) is symmetric, regular and satisfies the unitary relation. Moreover, when the relation (2.7) holds, the  $R$ -matrix (2.6) satisfies the Yang-Baxter equation:

$$R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2) R_{13}^{\uparrow\downarrow}(\lambda_1, \lambda_3) R_{23}^{\uparrow\downarrow}(\lambda_2, \lambda_3) = R_{23}^{\uparrow\downarrow}(\lambda_2, \lambda_3) R_{13}^{\uparrow\downarrow}(\lambda_1, \lambda_3) R_{12}^{\uparrow\downarrow}(\lambda_1, \lambda_2). \quad (2.8)$$

Being equipped with an  $R$ -matrix with all required properties, we can proceed to define the corresponding quantum integrable system, by performing the following steps: monodromy matrix, transfer matrix and Hamiltonian. The  $L$ -site monodromy matrix is given

$$T_{a\langle b_1 \dots b_L \rangle}(\lambda) = R_{ab_1}^{\uparrow\downarrow}(\lambda, 0) \dots R_{ab_L}^{\uparrow\downarrow}(\lambda, 0) \quad (2.9)$$

and its transfer matrix is the (super)trace in the auxiliary space:

$$t(\lambda) = \text{tr}_a T_{a\langle b_1 \dots b_L \rangle}(\lambda). \quad (2.10)$$

Then the generalized Hubbard Hamiltonian reads

$$H = \frac{d}{d\lambda} \ln t(\lambda) \Big|_{\lambda=0} = \sum_{x=1}^L H_{x,x+1} \quad (2.11)$$

with

$$H_{x,x+1} = (\Sigma P)_{\uparrow x, x+1} + (\Sigma P)_{\downarrow x, x+1} + u C_{\uparrow k} C_{\downarrow k}, \quad (2.12)$$

where we have used periodic boundary conditions. The notation  $\mathcal{O}_{\uparrow x, x+1}$  means that the operator  $\mathcal{O}$  acts non trivially in the parts  $x \uparrow$  and  $(x+1) \uparrow$  only. It acts as identity on all the sites different from  $x$  and  $x+1$  and also on the parts  $x \downarrow$  and  $(x+1) \downarrow$  of sites  $x$  and  $x+1$ . Explicitly, one has

$$(\Sigma P)_{\alpha x, x+1} = \sum_{j, \bar{j}} \left\{ (-1)^{[\bar{j}]} E_{\alpha x}^{j\bar{j}} E_{\alpha x+1}^{\bar{j}j} + (-1)^{[j]} E_{\alpha x}^{\bar{j}j} E_{\alpha x+1}^{j\bar{j}} \right\}$$

with  $\alpha = \uparrow$  or  $\downarrow$ . The indices  $j$  and  $\bar{j}$  run over  $\mathcal{N}$  and  $\overline{\mathcal{N}}$  respectively.

One can also define the momentum operator:

$$\exp(i\hat{p}) = t(0) = P_{1L} P_{2L} \dots P_{L-1, L}. \quad (2.13)$$

### 3 $gl(2|1) \oplus gl(2)$ model

#### 3.1 Preliminaries

We consider an example of the above model for a particular algebra  $gl(2|1)_{\uparrow} \oplus gl(2)_{\downarrow}$ . This notation means that for different spin value up or down we take different algebras in the construction of the Hamiltonian. The generic expression of the  $L$ -site Hamiltonian is still the same:

$$H_{gl(2|1) \oplus gl(2)} = \sum_{x=1}^L [(\Sigma P)_{\uparrow x, x+1} + (\Sigma P)_{\downarrow x, x+1}] + u \sum_{x=1}^L (C_{\uparrow x} C_{\downarrow x}), \quad (3.1)$$

but we now choose the projectors  $\pi_\uparrow$  and  $\pi_\downarrow$  to be such that

$$(\Sigma P)_{\uparrow, x, x+1} = E_{\uparrow x}^{12} E_{\uparrow x+1}^{21} + E_{\uparrow x}^{21} E_{\uparrow x+1}^{12} - E_{\uparrow x}^{13} E_{\uparrow x+1}^{31} + E_{\uparrow x}^{31} E_{\uparrow x+1}^{13}, \quad (3.2)$$

$$(\Sigma P)_{\downarrow, x, x+1} = E_{\downarrow x}^{12} E_{\downarrow x+1}^{21} + E_{\downarrow x}^{21} E_{\downarrow x+1}^{12}, \quad (3.3)$$

$$C_{\uparrow x} = E_{\uparrow x}^{11} - E_{\uparrow x}^{22} - E_{\uparrow x}^{33}, \quad C_{\downarrow x} = E_{\downarrow x}^{11} - E_{\downarrow x}^{22}. \quad (3.4)$$

Performing the Jordan-Wigner transformation one can write the Hamiltonian in terms of fermionic creation and annihilation operators (for more details see section 6):

$$\begin{aligned} H_{gl(2|1) \oplus gl(2)} = & \sum_{\substack{x=1 \\ \alpha=\uparrow, \downarrow}}^L \left\{ c_{\alpha x+1}^\dagger c_{\alpha x} + c_{\alpha x}^\dagger c_{\alpha x+1} \right\} + u \sum_{x=1}^L (1 - 2n_{\downarrow x}^c)(1 - 2n_{\uparrow x}^c) \\ & + \sum_{x=1}^L \left\{ (c_{\uparrow x+1}^\dagger c_{\uparrow x} + c_{\uparrow x}^\dagger c_{\uparrow x+1})(n_{\uparrow x}^d n_{\uparrow x+1}^d - n_{\uparrow x}^d - n_{\uparrow x+1}^d) \right. \\ & \quad \left. + c_{\uparrow x+1}^\dagger c_{\uparrow x} d_{\uparrow x+1}^\dagger d_{\uparrow x} + c_{\uparrow x}^\dagger c_{\uparrow x+1} d_{\uparrow x}^\dagger d_{\uparrow x+1} \right\} \\ & - u \sum_{x=1}^L (1 - 2n_{\downarrow x}^c)(1 - n_{\uparrow x}^c) n_{\uparrow x}^d \end{aligned} \quad (3.5)$$

where  $n_{\alpha x}^b = b_{\alpha x}^\dagger b_{\alpha x}$  is the particle number operator for  $b = c, d$ , and we assume standard relations between the operators ( $\alpha, \beta = \uparrow, \downarrow$ ):

$$\{c_{\alpha x}^\dagger, c_{\beta y}\} = \delta_{xy} \delta_{\alpha\beta}; \quad \{d_{\alpha x}^\dagger, d_{\beta y}\} = \delta_{xy} \delta_{\alpha\beta}; \quad \{c_{\alpha x}, d_{\beta y}\} = 0. \quad (3.6)$$

In the next subsection we present the system of Bethe equations corresponding to this Hamiltonian. Then, in the following subsections, we explain in details the method we used for our calculations: in subsection 3.3 we give the description of the coordinate Bethe ansatz approach for the first step of “nested”  $gl(2|1) \oplus gl(2)$  problem. Next subsections consist the explanations of the second and third steps of the problem.

### 3.2 Result for $gl(2|1) \oplus gl(2)$

As we shall see in the construction of the Bethe ansatz, the model describes three different kinds of “particles” ( $e^{2\uparrow}$ ,  $e^{2\downarrow}$  and  $e^{3\uparrow}$ ) above a “vacuum”. The particles  $e^{2\uparrow}$  and  $e^{2\downarrow}$  can be associated to spin up and down electrons, while  $e^{3\uparrow}$  represents a “spin 0” fermion (see section 6 for more details).

The energy of a state with  $N$  excitations is given by

$$E = L - 2N + 2 \sum_{m=1}^N \cos k_m \quad (3.7)$$

and its momentum reads

$$\mathbf{p} = \sum_{m=1}^N k_m \quad (3.8)$$

where the “impulsions” (or Bethe paramaters)  $k_m$  obey the Bethe equations of our  $gl(2|1) \oplus gl(2)$  model:

$$e^{ik_j L} = (-1)^{K+N+1} \prod_{m=1}^K \frac{i \sin k_j + ia_m + \frac{u}{4}}{i \sin k_j + ia_m - \frac{u}{4}}, \quad j = 1, \dots, N \quad (3.9)$$

$$(-1)^N \prod_{j=1}^N \frac{i \sin k_j + ia_m + \frac{u}{4}}{i \sin k_j + ia_m - \frac{u}{4}} = \Lambda(\vec{n}) \prod_{l=1, l \neq m}^K \frac{ia_m - ia_l + \frac{u}{2}}{ia_m - ia_l - \frac{u}{2}}, \quad m = 1, \dots, K \quad (3.10)$$

$$\Lambda(\vec{n}) = e^{\frac{2i\pi}{K}(n_1 + \dots + n_M)}, \quad M = 0, \dots, K, \quad 1 \leq n_1 < n_2 < \dots < n_M \leq K \quad (3.11)$$

where  $L$  is the number of sites considered in Hubbard model,  $N$  is total number of  $e^{2\uparrow}$ ,  $e^{2\downarrow}$  and  $e^{3\uparrow}$  “particles”.  $K$  counts the total number of excitations  $e^{2\uparrow}$  and  $e^{3\uparrow}$  and finally  $M$  numbers  $e^{3\uparrow}$  “particles”. The integers  $n_j$  correspond to the Bethe parameter of the last level, but their Bethe equation has been solved: it just corresponds to the quantization of these parameters (see details in section 4.4 that deals with a more general case).

### 3.3 Coordinate Bethe ansatz, level one

We use the coordinate Bethe ansatz to find the eigenvalues and eigenvectors of this model. In this model we have 5 types of distinct excitations noted  $e^{1\uparrow}, e^{2\uparrow}, e^{3\uparrow}$  and  $e^{1\downarrow}, e^{2\downarrow}$ . The objects  $e_x^{a\alpha}$  are elementary vectors (with 1 in  $a$  position and 0 elsewhere) which form a natural basis for the vector space  $V_{\alpha x} \sim \mathbb{C}^5$  ( $\alpha$  and  $x$  hold to describe spin and position in physical space). First of all we define the reference state (pseudo-vacuum):

$$\phi_0 = \prod_{x=1}^L e_x^{1\uparrow} e_x^{1\downarrow}. \quad (3.12)$$

In the following calculations we use the expression of the Hamiltonian only in matrix form (3.1). The corresponding eigenvalue is given by:

$$H_{gl(2|1) \oplus gl(2)} \phi_0 = uL \phi_0. \quad (3.13)$$

We define an excitation by a pair  $(A, \alpha)$ ,  $A = 2$  or  $3$  (corresponding to vectors  $e^2$  or  $e^3$ ) and  $\alpha = \uparrow, \downarrow$  and there is no pair  $(3, \downarrow)$ . The  $N$  excitation states of the Hamiltonian can be written as

$$\phi[(\bar{A}, \bar{\alpha})] = \sum_{\mathbf{x}} \Psi[\mathbf{x}, (\bar{A}, \bar{\alpha})] e_{x_1}^{A_1 \alpha_1} \dots e_{x_N}^{A_N \alpha_N} \quad (3.14)$$

where the sum over  $\mathbf{x} = (x_1, x_2, \dots, x_N)$  is considered without coinciding points  $x_l = x_m$  such that  $\alpha_l = \alpha_m$  for any  $l, m = 1, \dots, N$  (this describes the exclusion principle for identical particles). We also use the notation  $(\bar{A}, \bar{\alpha}) = (A_1, \alpha_1) \dots (A_N, \alpha_N)$ , and we omitted the ‘empty sites’, i.e. sites carrying vectors  $e^{1\uparrow} e^{1\downarrow}$ .

Applying the Hamiltonian to (3.14), one gets the eigenvalue equation for the  $\Psi$  function, given by

$$\begin{aligned} & \sum_{m=1}^N (\Psi[\mathbf{x} - \mathbf{e}_m, (\bar{A}, \bar{\alpha})] \Delta_m^- + \Psi[\mathbf{x} + \mathbf{e}_m, (\bar{A}, \bar{\alpha})] \Delta_m^+) + \\ & + \left( (L - 2N) + u \sum_{l,n} \delta(x_l - x_n) \delta(\alpha_l \neq \alpha_n) - E \right) \Psi[\mathbf{x}, (\bar{A}, \bar{\alpha})] \Delta^3 = 0 \end{aligned} \quad (3.15)$$



where  $\mathbf{e}_m$  is an elementary vector in  $\mathbb{C}^N$  with entry 1 on the  $m^{\text{th}}$  position and 0 elsewhere. Also we denoted

$$\Delta_m^\pm = \prod_{l \neq m} \prod_{n \neq m} \delta^\dagger(x_l \neq x_n) \prod_l \delta^\dagger(x_l \neq x_m) \delta^\dagger(x_l \neq x_m \pm 1), \quad (3.16)$$

$$\Delta^3 = \prod_{l \neq n} \delta^\dagger(x_l \neq x_n), \quad (3.17)$$

$$\delta^\dagger(x_l \neq x_n) = 1 - \delta(x_l - x_n) \delta(\alpha_l - \alpha_n). \quad (3.18)$$

All these symbols mean that there is no particle with the same spin on the same and neighbouring sites with some conditions corresponding to each symbol (exclusion principle).

It is convenient in the following to denote  $\Psi[\mathbf{x}, (\bar{A}, \bar{\alpha})]$  simply by  $\Psi(\mathbf{x})$  when there is no ambiguity.

At the first step, we take a non-interacting regime, which implies  $x_1 \ll x_2 \ll \dots \ll x_N$  (in other words, the particles are far enough from each other). As a consequence, all symbols in (3.18) are equal to 1 and there is no interaction term. We look for a solution of this equation in a form of “free particles”, namely  $\Psi(\mathbf{x}) \propto e^{i\mathbf{k}\mathbf{x}}$ , where  $\{k_1, k_2, \dots, k_N\}$  denote a set of unequal numbers.

We get therefore the value of energy

$$E = L - 2N + 2 \sum_{m=1}^N \cos k_m. \quad (3.19)$$

In order to consider all other cases of the particle dispositions, we assume the *Bethe hypothesis* for the general solution of  $\Psi(\mathbf{x})$ . We divide the coordinate space  $(x_1, x_2, \dots, x_N)$  into  $N!$  sectors: for  $x_{q_1} < x_{q_2} < \dots < x_{q_N}$ ,

$$\Psi_Q(\mathbf{x}) = \sum_P (-1)^{[sg(Q)]} \Phi(P, QP^{-1}) e^{iP\mathbf{k} \cdot Q\mathbf{x}} \quad (3.20)$$

where  $P = [p_1, p_2, \dots, p_N]$  and  $Q = [q_1, q_2, \dots, q_N]$  are two elements of the permutation group  $\mathfrak{S}_N$  and  $P\mathbf{k} \cdot Q\mathbf{x} = \sum_i k_{p_i} x_{q_i}$ . The symbol  $(-1)^{[sg(Q)]}$  stands for the signature of the  $Q$ -permutation when restricted to grade 1 particles (that is  $e^{3\uparrow}$ ). For instance, we have the property (valid for any permutation  $Q$  and any permutation  $\Pi_{ii+1}$ ):  $(-1)^{[sg(Q\Pi_{ii+1})]} = (-1)^{[sg(Q)] + [A_i][A_{i+1}]}$ . We recall that  $\Psi(\mathbf{x})$  and accordingly  $\Phi(P, QP^{-1})$  depend on the type of excitations  $(\bar{A}, \bar{\alpha})$ .

The coefficients  $\Phi(P, QP^{-1})$  are not all independent and the application of the Hamiltonian represented in equation (3.15) can reduce their number in several cases.

1. Let us consider the sector  $x_{q_1} \ll \dots \ll x_{q_i} < x_{q_{i+1}} \ll x_{q_N}$  with  $x_{q_i} = x_{q_{i+1}} - 1$  and  $\alpha_{q_i} = \alpha_{q_{i+1}}$ . In this case  $\Delta_{q_{i+1}}^- = 0$  and  $\Delta_{q_i}^+ = 0$ , all other symbols in (3.18) are equal to 1. Thus, equation (3.15) becomes

$$\sum_{m \neq q_i, q_{i+1}} \left( \Psi_Q(\mathbf{x} - \mathbf{e}_m) + \Psi_Q(\mathbf{x} + \mathbf{e}_m) \right) + \Psi_Q(\mathbf{x} - \mathbf{e}_{q_i}) + \Psi_Q(\mathbf{x} + \mathbf{e}_{q_{i+1}}) = (E - L + 2N) \Psi_Q(\mathbf{x}) \quad (3.21)$$

Using (3.20) and performing simple algebraic calculation we obtain

$$\Phi(\Pi_{ab}P, QP^{-1}\Pi_{ab}) = -\Phi(P, QP^{-1}), \quad (3.22)$$

where  $\Pi_{ab}$  is the permutation of objects  $a$  and  $b$  which are linked to  $i, i+1$  by  $p^{-1}(a) = i$  and  $p^{-1}(b) = i+1$ . This relation holds for any value of  $A_{q_i}$  and  $A_{q_{i+1}}$ .

2. Now let us consider another case:  $x_{q_1} \ll \dots \ll x_{q_i} = x_{q_{i+1}} \ll x_{q_N}$  and  $\alpha_{q_i} \neq \alpha_{q_{i+1}}$ . We denote by  $\bar{Q}$  the sector where  $i$  and  $i+1$  are permuted (i.e.  $\bar{Q} = Q\Pi_{ii+1}$ ). It implies that the equation (3.15) becomes

$$\begin{aligned} \sum_{m \neq q_i, q_{i+1}} [\Psi_Q(\mathbf{x} - \mathbf{e}_m) + \Psi_Q(\mathbf{x} + \mathbf{e}_m)] + \Psi_Q(\mathbf{x} - \mathbf{e}_{q_i}) + \Psi_{\bar{Q}}(\mathbf{x} + \mathbf{e}_{q_i}) + \\ + \Psi_{\bar{Q}}(\mathbf{x} - \mathbf{e}_{q_{i+1}}) + \Psi_Q(\mathbf{x} + \mathbf{e}_{q_{i+1}}) + ((L - 2N) - E) \Psi_Q(\mathbf{x}) = 0. \end{aligned} \quad (3.23)$$

On the other hand, we have the condition of continuity of the wave function

$$\Psi_Q(\mathbf{x}) = \Psi_{\bar{Q}}(\mathbf{x}) \quad \text{with} \quad x_{q_i} = x_{q_{i+1}} \quad (3.24)$$

These last two equations together give two conditions on the coefficients  $\Phi(P, QP^{-1})$  which can be written in matrix form

$$\begin{pmatrix} \Phi(\Pi_{ab}P, QP^{-1}) \\ \Phi(\Pi_{ab}P, QP^{-1}\Pi_{ab}) \end{pmatrix} = \begin{pmatrix} t_{ab} & r_{ab} \\ r_{ab} & t_{ab} \end{pmatrix} \begin{pmatrix} \Phi(P, QP^{-1}) \\ \Phi(P, QP^{-1}\Pi_{ab}) \end{pmatrix} \quad (3.25)$$

with

$$t_{ab} = \frac{2i(\lambda_a - \lambda_b)}{u + 2i(\lambda_a - \lambda_b)}, \quad r_{ab} = \frac{-u}{u + 2i(\lambda_a - \lambda_b)}. \quad (3.26)$$

These equations also hold for any type of excitations (any value of  $A_{q_i}$  and  $A_{q_{i+1}}$ ). We set  $\lambda_a = \sin k_a$  to simplify the expressions.

Before applying the periodic conditions on the coefficients  $\Phi(P, QP^{-1})$  (henceforth we will rename  $QP^{-1}$  simply by  $Q$ ), we rewrite equations (3.22) and (3.25) in a compact form. Let us introduce the vector

$$\hat{\Phi}(P) \equiv \sum_{Q, (\bar{A}, \bar{\alpha})} \Phi[P, Q, (\bar{A}, \bar{\alpha})] |(A_{q_1}, \alpha_{q_1}), \dots, (A_{q_N}, \alpha_{q_N}) \rangle \quad (3.27)$$

where the sum is over all types of excitations and all corresponding sectors. The vector  $|(A_{q_1}, \alpha_{q_1}), \dots, (A_{q_N}, \alpha_{q_N}) \rangle$  belongs to  $V_1 \otimes \dots \otimes V_N$ , where  $V = \text{span}\{2 \uparrow, 2 \downarrow, 3 \uparrow\}$ , and represents one state in the space of  $N$  excitations. Let us illustrate by an example for  $N = 2$  excitations: in this case we have 6 types of different excitations.

$$\begin{aligned} \sum_{Q, (\bar{A}, \bar{\alpha})} \Phi[P, Q, (\bar{A}, \bar{\alpha})] |(A_{q_1}, \alpha_{q_1}), (A_{q_2}, \alpha_{q_2}) \rangle = & \Phi(P, id, 1) |2 \uparrow, 2 \uparrow \rangle + \Phi(P, id, 2) |2 \uparrow, 2 \downarrow \rangle \\ & + \Phi(P, id, 3) |2 \uparrow, 3 \uparrow \rangle + \Phi(P, \Pi_{12}, 2) |2 \downarrow, 2 \uparrow \rangle + \Phi(P, id, 4) |2 \downarrow, 2 \downarrow \rangle + \\ & + \Phi(P, id, 5) |2 \downarrow, 3 \uparrow \rangle + \Phi(P, \Pi_{12}, 3) |3 \uparrow, 2 \uparrow \rangle + \Phi(P, \Pi_{12}, 5) |3 \uparrow, 2 \downarrow \rangle \\ & + \Phi(P, id, 6) |3 \uparrow, 3 \uparrow \rangle. \end{aligned} \quad (3.28)$$

Thus, for  $N = 2$  excitations we can express (3.22) and (3.25) by

$$\begin{aligned}
 \hat{\Phi}(\Pi_{12}P) &= (-1)\Phi(P, id, 1)|2 \uparrow, 2 \uparrow\rangle + [t_{12}\Phi(P, id, 2) + r_{12}\Phi(P, \Pi_{12}, 2)]|2 \uparrow, 2 \downarrow\rangle \\
 &\quad + [t_{12}\Phi(P, \Pi_{12}, 2) + r_{12}\Phi(P, id, 2)]|2 \downarrow, 2 \uparrow\rangle + (-1)\Phi(P, id, 3)|2 \uparrow, 3 \uparrow\rangle \\
 &\quad + (-1)\Phi(P, \Pi_{12}, 3)|3 \uparrow, 2 \uparrow\rangle + \dots + (-1)\Phi(P, id, 6)|3 \uparrow, 3 \uparrow\rangle \\
 &\equiv S_{12}^{(1)}(\lambda_1 - \lambda_2)\hat{\Phi}(P)
 \end{aligned} \tag{3.29}$$

where we introduced the so-called  $S$ -matrix  $S_{12}^{(1)}(\lambda)$ :

$$S_{12}^{(1)}(\lambda_1 - \lambda_2) \begin{pmatrix} |2 \uparrow, 2 \uparrow\rangle \\ |2 \uparrow, 2 \downarrow\rangle \\ |2 \uparrow, 3 \uparrow\rangle \\ |2 \downarrow, 2 \uparrow\rangle \\ |2 \downarrow, 2 \downarrow\rangle \\ |2 \downarrow, 3 \uparrow\rangle \\ |3 \uparrow, 2 \uparrow\rangle \\ |3 \uparrow, 2 \downarrow\rangle \\ |3 \uparrow, 3 \uparrow\rangle \end{pmatrix} = \begin{pmatrix} -1 & & & & & & & & & \\ & t_{12} & r_{12} & & & & & & & \\ & & 0 & & & -1 & & & & \\ \hline & r_{12} & t_{12} & & & & & & & \\ & & & -1 & & & & & & \\ & & & & t_{12} & r_{12} & & & & \\ \hline & & -1 & & & 0 & & & & \\ & & & & r_{12} & t_{12} & & & & \\ & & & & & & & & -1 & \end{pmatrix} \begin{pmatrix} |2 \uparrow, 2 \uparrow\rangle \\ |2 \uparrow, 2 \downarrow\rangle \\ |2 \uparrow, 3 \uparrow\rangle \\ |2 \downarrow, 2 \uparrow\rangle \\ |2 \downarrow, 2 \downarrow\rangle \\ |2 \downarrow, 3 \uparrow\rangle \\ |3 \uparrow, 2 \uparrow\rangle \\ |3 \uparrow, 2 \downarrow\rangle \\ |3 \uparrow, 3 \uparrow\rangle \end{pmatrix} \tag{3.30}$$

The coefficients  $t_{12}$  and  $r_{12}$  are defined in (3.26). The same reasoning could be repeated for an arbitrary number  $N$  of excitations, and we obtain

$$\hat{\Phi}(\Pi_{ab}P) = S_{ab}^{(1)}(\lambda_a - \lambda_b)\hat{\Phi}(P), \tag{3.31}$$

where the matrix  $S_{ab}^{(1)}(\lambda_a - \lambda_b)$  acts nontrivially only on vector spaces  $V_a \otimes V_b$ . It satisfies the Yang-Baxter equation:

$$S_{12}^{(1)}(\lambda_1 - \lambda_2)S_{13}^{(1)}(\lambda_1 - \lambda_3)S_{23}^{(1)}(\lambda_2 - \lambda_3) = S_{23}^{(1)}(\lambda_2 - \lambda_3)S_{13}^{(1)}(\lambda_1 - \lambda_3)S_{12}^{(1)}(\lambda_1 - \lambda_2). \tag{3.32}$$

3. In order to obtain the Bethe equations we should apply the periodic boundary conditions on the function  $\Psi(\mathbf{x})$ . Let  $C$  be the cyclic permutation given by  $C = \Pi_{N1} \dots \Pi_{NN-1}$ . More precisely the periodicity condition means

$$\Psi_{QC}(\mathbf{x} + \mathbf{e}_{q_1}L) = \Psi_Q(\mathbf{x}) \tag{3.33}$$

which implies a condition on the coefficients  $\Phi(P, Q)$ , namely

$$\hat{\Phi}(PC) = e^{ik_{p_N}L}\hat{\Phi}(P) \tag{3.34}$$

choosing  $P = C^{N-j}$ , one derives a system of equations satisfied by the coefficients  $\hat{\Phi}(id)$  which is called ‘‘auxiliary problem’’. For  $j = 1, \dots, N$ , it reads

$$S_{j+1j}^{(1)} \dots S_{Nj}^{(1)} S_{1j}^{(1)} \dots S_{j-1j}^{(1)} \hat{\Phi}(id) = e^{ik_j L} \hat{\Phi}(id) \tag{3.35}$$

here we omitted the arguments in  $S$ -matrices,  $S_{ab}^{(1)} \equiv S_{ab}^{(1)}(\lambda_a - \lambda_b)$ .

If we could perform the diagonalization of the left-hand-side in the general case, it would be possible right now to write the Bethe equations for our model. The form of matrix  $S_{ab}^{(1)}$  requires to use again the coordinate Bethe ansatz to solve the auxiliary problem.

### 3.4 Auxiliary problem, level two

Again we have the equation on eigenvectors and eigenvalues to solve, but contrarily to the first step the Hamiltonian is a little more complicated:

$$S_{j+1j}^{(1)} \dots S_{Nj}^{(1)} S_{1j}^{(1)} \dots S_{j-1j}^{(1)} \phi = \Lambda_j \phi \quad (3.36)$$

Since the matrix  $S^{(1)}$  is regular, the above Hamiltonian can be identified with  $\tilde{t}(\lambda_j)$ , where we have introduced a new transfer matrix

$$\tilde{t}(\lambda) = \text{tr}_0 \left( S_{j+1,0}^{(1)}(\lambda_{j+1} - \lambda) \dots S_{N,0}^{(1)}(\lambda_N - \lambda) S_{1,0}^{(1)}(\lambda_1 - \lambda) \dots S_{j-1,0}^{(1)}(\lambda_{j-1} - \lambda) S_{j,0}^{(1)}(\lambda_j - \lambda) \right)$$

Since  $S^{(1)}$  obeys Yang-Baxter equation, this new problem is indeed integrable. We can thus perform the Bethe ansatz again.

First, we slightly modify the  $S$ -matrix  $S_{12}^{(1)} \rightarrow -S_{12}^{(1)} \equiv S_{12}$  to simplify the following calculations. Thus, it is given by

$$S_{12}(\lambda_1 - \lambda_2) = P_{12} - t_{12} K_{12}, \quad \text{with} \quad t_{12} = \frac{i(\lambda_1 - \lambda_2)}{i(\lambda_1 - \lambda_2) + \frac{u}{2}}, \quad (3.37)$$

$$P_{12} = \sum_{i,j=1,2,3} E^{ij} \otimes E^{ji}, \quad (3.38)$$

$$K_{12} = \sum_{i=2,3} (E^{11} \otimes E^{ii} + E^{ii} \otimes E^{11} + E^{1i} \otimes E^{i1} + E^{i1} \otimes E^{1i}). \quad (3.39)$$

We use the coordinate Bethe ansatz to proceed with this problem. At this step we have only three types of different excitations:  $e^{2\uparrow}$ ,  $e^{2\downarrow}$  and  $e^{3\uparrow}$ . At first, we choose the reference state (pseudovacuum) to be

$$\phi_0 = \prod_{k=1}^N e_k^{2\downarrow},$$

with eigenvalue  $\Lambda = 1$ .

Applying the  $S$ -matrix on different states (on the pseudovacuum  $e^2$  and two excitations  $e^1$  and  $e^3$ ), we get

$$S_{12}(\lambda_1 - \lambda_2) e^{2\downarrow} \otimes e^{2\downarrow} = e^{2\downarrow} \otimes e^{2\downarrow}, \quad (3.40)$$

$$S_{12}(\lambda_1 - \lambda_2) e^A \otimes e^{2\downarrow} = -t_{12} e^A \otimes e^{2\downarrow} + (1 - t_{12}) e^{2\downarrow} \otimes e^A, \quad (3.41)$$

$$S_{12}(\lambda_1 - \lambda_2) e^{2\downarrow} \otimes e^A = -t_{12} e^{2\downarrow} \otimes e^A + (1 - t_{12}) e^A \otimes e^{2\downarrow}, \quad (3.42)$$

$$S_{12}(\lambda_1 - \lambda_2) e^A \otimes e^B = e^B \otimes e^A \quad \text{with} \quad A, B = 2\uparrow, 3\uparrow. \quad (3.43)$$

We first consider the “non-interacting” regime, which implies to solve the equation with only one excitation. Let  $A = 2\uparrow$  or  $3\uparrow$  and indicate the type of the excitation

$$\phi(A) = \sum_x f_{x,A} e_x^A. \quad (3.44)$$

We will forget henceforth the  $A$  index to simplify the notations.

Here, contrarily to the first subsection, it is not obvious to readily calculate the Hamiltonian action on excited states. We can do it step by step introducing a recursive Hamiltonian (for details, see [19])

$$S_{j-kj} \dots S_{j-1j} \phi = \Lambda_j \phi \tag{3.45}$$

and recursive coefficients  $f_x^{(k)}$  which represent the action of the recursive Hamiltonian on the coefficients  $f_x$ . One gets the following relations

$$\begin{aligned} \sum_x f_x^{(1)} e_x^A &\equiv S_{j-1j} \sum_x f_x e_x^A, \\ &\vdots \\ \sum_x f_x^{(k)} e_x^A &\equiv S_{j-kj} \dots S_{j-1j} \sum_x f_x e_x^A. \end{aligned} \tag{3.46}$$

We write down the recursive relations between coefficients  $f_x^{(k)}$  and  $f_x^{(k-1)}$ . For  $x \neq j, j-k$

$$\begin{aligned} f_x^{(k)} &= f_x^{(k-1)}, \\ f_j^{(k)} &= -t_{j-kj} f_j^{(k-1)} + (1 - t_{j-kj}) f_{j-k}^{(k-1)}, \\ f_{j-k}^{(k)} &= -t_{j-kj} f_{j-k}^{(k-1)} + (1 - t_{j-kj}) f_j^{(k-1)}. \end{aligned} \tag{3.47}$$

Skipping the details of calculation, the solution of the equation (3.45) gives a relation between different  $f_x$ . Closer considering shows that one needs to introduce an additional constant  $a$  (particle rapidity), such that

$$\frac{f_{x+1}(a)}{f_x(a)} = -\frac{i\lambda_x + ia + \frac{u}{4}}{i\lambda_{x+1} + ia - \frac{u}{4}}. \tag{3.48}$$

Thus, we can write down the expression for coefficient  $f_x$  and the eigenvalue corresponding to one excitation. We normalize  $f_1 = 1$ , then we have

$$f_x(a) = \prod_{m=1}^{x-1} \left( -\frac{i\lambda_m + ia + \frac{u}{4}}{i\lambda_{m+1} + ia - \frac{u}{4}} \right), \tag{3.49}$$

$$\Lambda_j \equiv \sigma_j(a) = -\frac{i\lambda_j + ia + \frac{u}{4}}{i\lambda_j + ia - \frac{u}{4}}. \tag{3.50}$$

It is convenient to write down some relation between iterated coefficient  $f_x^{(k)}$ ,  $f_x$  and  $\sigma$ :

$$\begin{aligned} f_x(a) &= \sigma_1(a) \dots \sigma_{x-1}(a) \frac{i\lambda_1 + ia - \frac{u}{4}}{i\lambda_x + ia - \frac{u}{4}}, \\ f_j^{(k)}(a) &= \frac{f_j(a)}{\sigma_{j-1}(a) \dots \sigma_{j-k}(a)}, \\ f_{j-k}^{(k)}(a) &= \sigma_j(a) f_{j-k}(a). \end{aligned} \tag{3.51}$$

Starting from the initial equation (3.36), we can choose  $j = N$  and without any loss of generality we can derive the equation which represents the Bethe equations on rapidity  $a$  by

application of the initial Hamiltonian on  $f_x(a)$ . It also automatically implies the periodic boundary condition

$$\prod_{m=1}^N \sigma_m(a) = 1. \quad (3.52)$$

All these results hold for any type of excitation  $e^{2\uparrow}$  or  $e^{3\uparrow}$  (we neglected the index A above).

Now we can proceed to the case of K excitations. Let  $\bar{A}$  be a vector  $(A_1, \dots, A_K)$  with  $A_i = 2\uparrow, 3\uparrow$ , which represents the combination of different excitations in consideration. The eigenvector is naturally represented as

$$\phi(\bar{A}) = \sum_{\mathbf{x}} \Psi(\mathbf{x}, \bar{A}) e_{x_1}^{A_1} \dots e_{x_K}^{A_K}. \quad (3.53)$$

The sum is done over all coordinates  $x_i$  without coinciding points  $x_i = x_k$  for any  $i, k$  (exclusion principle).

We use the Bethe hypothesis for the coefficients  $\Psi(\mathbf{x}, \bar{A})$ , in the sector  $x_{q_1} < x_{q_2} < \dots < x_{q_K}$  where  $Q = [q_1, \dots, q_K]$  is a permutation of the integers  $1, 2, \dots, K$ :

$$\Psi(\mathbf{x}, \bar{A}) = \sum_P \Phi(P, QP^{-1}, \bar{A}) f_{x_{q_1}}(a_{p_1}) f_{x_{q_2}}(a_{p_2}) \dots f_{x_{q_K}}(a_{p_K}) \quad (3.54)$$

where  $f_{x_{q_i}}(a_{p_i})$  is the one-particle solution with rapidity  $a_{p_i}$ . We will omit the index  $\bar{A}$  to simplify the notations. As above, one can reduce the number of coefficients  $\Phi(P, QP^{-1})$ .

At this stage we will consider K excitations of different types. Therefore we have  $K!/M!(K-M)!$  sectors  $x_{q_1} < x_{q_2} < \dots < x_{q_K}$ , where  $M, K-M$  are the numbers of excitation of type  $2\uparrow$  and  $3\uparrow$  respectively. Acting with the recursive Hamiltonian on the eigenvector  $\phi$ , see (3.45), we are able to write down the relations between different  $\Psi_Q^{(k)}(\mathbf{x})$ .

We consider  $j$  to take the biggest value such that we do not have any excitation on the sites bigger than  $j$ . Several cases can occur, depending on whether there is an excitation on sites  $j$  and/or  $j-k$ . If there is an excitation on site  $j$ , it must correspond to  $x_{q_K} = j$ . For the site  $j-k$ , there exists  $q_m$  such that we have either  $x_{q_m} < j-k < x_{q_{m+1}}$  (no excitation at site  $j-k$ ), or  $x_{q_m} = j-k$  (one excitation at site  $j-k$ ). Then, we have

$$\begin{aligned} \Psi_Q^{(k)}(\mathbf{x}) &= \Psi_Q^{(k-1)}(\mathbf{x}), \quad \text{for } \mathbf{x} \neq j, j-k, \\ \Psi_Q^{(k)}(\dots, \downarrow_j^{x_{q_K}}, \dots) &= -t_{j-kj} \Psi_Q^{(k-1)}(\dots, \downarrow_j^{x_{q_K}}, \dots) + (1-t_{j-kj}) \Psi_{Q'}^{(k-1)}(\dots, \downarrow_{j-k}^{x_{q_K}}, \dots), \\ &\quad \text{with } Q' = Q \Pi_{K m+1} \dots \Pi_{K K-1} \quad \text{for } \mathbf{x} \neq j-k, \\ \Psi_Q^{(k)}(\dots, \downarrow_{j-k}^{x_{q_m}}, \dots) &= -t_{j-kj} \Psi_Q^{(k-1)}(\dots, \downarrow_{j-k}^{x_{q_m}}, \dots) + (1-t_{j-kj}) \Psi_{Q'}^{(k-1)}(\dots, \downarrow_j^{x_{q_m}}, \dots) \\ &\quad \text{with } Q' = Q \Pi_{m K} \dots \Pi_{m m+1} \quad \text{for } \mathbf{x} \neq j \end{aligned} \quad (3.55)$$

In this last case, we have also the exclusion principle: for any  $n \neq m$ ,  $x_n \neq j, j-k$ .

When two excitations are on sites  $j$  and  $j-k$ , with  $x_{q_K} = j$  and  $x_{q_m} = j-k$ , we have:

$$\begin{aligned} \Psi_Q^{(k)}(\dots, j-k, \dots, j, \dots) &= \Psi_{Q \Pi_{K m}}^{(k-1)}(\dots, j, \dots, j-k, \dots) \quad \text{for non-identical excitations,} \\ \Psi_Q^{(k)}(\dots, j-k, \dots, j, \dots) &= \Psi_Q^{(k-1)}(\dots, j-k, \dots, j, \dots) \quad \text{if the excitations are identical.} \end{aligned}$$

On the one hand, we have the Bethe hypothesis for the coefficients  $\Psi(\mathbf{x})$  composed of the one-excitation functions  $f_{x_i}$ ; on the other hand, we have the relations for the iterated coefficients, it is natural to calculate how the iteration passes to the one-excitation functions. In other words, we want to calculate the coefficients  $\Psi_Q^{(k)}(\mathbf{x})$  in terms of the iterated free excitation functions  $f_x^{(k)}(a_p)$ .

After first iteration for any  $\mathbf{x}$  except the coefficient  $\Psi_Q^{(1)}(\dots, j-1, \dots, j, \dots)$  we find

$$\Psi_Q^{(1)}(\mathbf{x}) = \sum_P \Phi(P, QP^{-1}) f_{x_{q_1}}^{(1)}(a_{p_1}) f_{x_{q_2}}^{(1)}(a_{p_2}) \dots f_{x_{q_K}}^{(1)}(a_{p_K}). \quad (3.56)$$

The next iteration will allow us to make an assumption on the form of  $\Psi_Q^{(1)}(\dots, j-1, \dots, j, \dots)$ . Thus we calculate the coefficients  $\Psi_Q^{(2)}(\mathbf{x})$  except again those where two coordinates coincide with  $j-2$  and  $j$ . Skipping the calculations we find for the first iteration

$$\Psi_Q^{(1)}(\dots, j-1, \dots, j, \dots) = \sum_P \Phi(P, (Q\Pi_{KK-1})P^{-1}) f_{x_{q_1}}^{(1)}(a_{p_1}) \dots f_j^{(1)}(a_{p_{K-1}}) f_{j-1}^{(1)}(a_{p_K}).$$

This implies the conditions

$$\Psi_Q^{(1)}(\dots, j-1, j, \dots) = \Psi_Q(\dots, j-1, j, \dots) \quad \text{for identical excitations} \quad (3.57)$$

$$\Psi_Q^{(1)}(\dots, j-1, j, \dots) = \Psi_{Q'}(\dots, j, j-1, \dots) \quad \text{for different type of excitations} \quad (3.58)$$

with  $Q' = Q\Pi_{KK-1}$ . It is useful to notice that for identical excitations  $Q\Pi_{KK-1}$  defines the same sector as  $Q$ . These equations give the connection between  $\Phi(P\Pi_{KK-1}, Q(P\Pi_{KK-1})^{-1})$  and  $\Phi(P, QP^{-1})$  for some sectors  $Q$ . We will consider it precisely in the general case.

For the second iteration we find

$$\Psi_Q^{(2)}(\mathbf{x}) = \sum_P \Phi(P, QP^{-1}) f_{x_{q_1}}^{(2)}(a_{p_1}) \dots f_{x_{q_K}}^{(2)}(a_{p_K}) \quad (3.59)$$

except

$$\Psi_Q^{(2)}(\dots, j-1, \dots, j, \dots) = \sum_P \Phi(P, (Q\Pi_{KK-1})P^{-1}) f_{x_{q_1}}^{(2)}(a_{p_1}) \dots f_j^{(2)}(a_{p_{K-1}}) f_{j-1}^{(2)}(a_{p_K}). \quad (3.60)$$

and as in the previous case we have two ‘‘undefined’’ coefficients  $\Psi^{(2)}(\dots, j-2, \dots, j, \dots)$  and  $\Psi^{(2)}(\dots, j-2, \dots, j-1, \dots, j, \dots)$ .

Generalizing these results until the  $(k+1)$ -th iteration, we make again a set of assumptions for the coefficients of type  $\Psi^{(k)}(\dots, j-k, \dots, j, \dots)$ . Hence, for the  $k$ -th iteration, we find the coefficients

$$\Psi_Q^{(k)}(\mathbf{x}) = \sum_P \Phi(P, QP^{-1}) f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_{x_{q_K}}^{(k)}(a_{p_K}) \quad (3.61)$$

except a set of coefficients. Let  $n_l$  be integers such that  $0 < n_1 < n_2 < \dots < n_l < k$ ,  $l = 1, \dots, k-1$ , that label the position of the possible excitations between  $j-k$  and  $j$ . We have a set of assumptions made from  $(k+1)$ -th iteration.<sup>1</sup>

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<sup>1</sup>In order to make the notation shorter we do not write the dots ... for unchanged indices when it is not ambiguous.

If there is an excitation in position  $j - k$ , we have:

$$\Psi_Q^{(k)}(j - k, j) = \sum_P \Phi(P, (Q\Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-1}})f_{j-k}^{(k)}(a_{p_K}) \quad (3.62)$$

$$\Psi_Q^{(k)}(j - k, j - n_1, j) = \quad (3.63)$$

$$\sum_P \Phi(P, (Q\Pi_K^{K-2}\Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-2}})f_{j-k}^{(k)}(a_{p_{K-1}})f_{j-n_1}^{(k)}(a_{p_K})$$

$\vdots$

$$\Psi_Q^{(k)}(j - k, j - n_l, \dots, j - n_1, j) = \quad (3.64)$$

$$\sum_P \Phi(P, (Q\Pi_K^{K-l-1} \dots \Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-l-1}})f_{j-k}^{(k)}(a_{p_{K-l}}) \dots f_{j-n_1}^{(k)}(a_{p_K})$$

$\vdots$

$$\Psi_Q^{(k)}(j - k, j - k + 1, \dots, j - 1, j) = \quad (3.65)$$

$$\sum_P \Phi(P, (Q\Pi_K^{K-k} \dots \Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-k}})f_{j-k}^{(k)}(a_{p_{K-k+1}}) \dots f_{j-1}^{(k)}(a_{p_K})$$

If there is no  $x$  equal to  $j - k$ , from the calculation we have

$$\Psi_Q^{(k)}(j - n_1, j) = \sum_P \Phi(P, (Q\Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-1}})f_{j-n_1}^{(k)}(a_{p_K}), \quad (3.66)$$

$$\Psi_Q^{(k)}(j - n_2, j - n_1, j) =$$

$$\sum_P \Phi(P, (Q\Pi_K^{K-2}\Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-2}})f_{j-n_2}^{(k)}(a_{p_{K-1}})f_{j-n_1}^{(k)}(a_{p_K}), \quad (3.67)$$

$\vdots$

$$\Psi_Q^{(k)}(j - k + 1, \dots, j - 1, j) = \quad (3.68)$$

$$\sum_P \Phi(P, (Q\Pi_K^{K-k+1} \dots \Pi_K^{K-1})P^{-1})f_{x_{q_1}}^{(k)}(a_{p_1}) \dots f_j^{(k)}(a_{p_{K-k+1}})f_{j-k+1}^{(k)}(a_{p_{K-k+2}}) \dots f_{j-1}^{(k)}(a_{p_K})$$

We can write a set of conditions from all made assumptions to determine the relations between different  $\Phi(P, P^{-1})$ . One can see that we can consider only the  $k$ -th iteration to derive the necessary relations. More exactly we have  $k$  equations to satisfy

$$\Psi_Q^{(k)}(j - k, j - n_l, \dots, j - n_1, j) = \Psi_Q^{(k-1)}(j - k, j - n_l, \dots, j - n_1, j) \quad (3.69)$$

for identical excitations corresponding to  $j$  and  $j - k$ . Remark that in this case  $Q\Pi_K^{K-l-1}$  is identical to  $Q$ .

For non-identical excitations, we have

$$\Psi_Q^{(k)}(j - k, j - n_l, \dots, j - n_1, j) = \Psi_{Q\Pi_K^{K-l-1}}^{(k-1)}(j, j - n_l, \dots, j - n_1, j - k) \quad (3.70)$$

with  $l = 0, \dots, k - 1$  in both cases. Hence we have a set of conditions:

$$\begin{aligned} & \sum_P \Phi(P, (Q\Pi_K^{K-l-1} \dots \Pi_K^{K-1})P^{-1})f^{(k)} \dots f_j^{(k)}(a_{p_{K-l-1}})f_{j-k}^{(k)}(a_{p_{K-l}}) \dots f_{x_{j-n_1}}^{(k)}(a_{p_K}) = \\ & = \sum_P \Phi(P, (Q\Pi_K^{K-l-1} \dots \Pi_K^{K-1})P^{-1})f^{(k-1)} \dots f_{j-k}^{(k-1)}(a_{p_{K-l-1}})f_j^{(k-1)}(a_{p_{K-l}}) \dots f_{x_{j-n_1}}^{(k-1)}(a_{p_K}). \end{aligned}$$



Using the expressions obtained for iterated one-particle functions (3.51) and omitting all the algebraic calculations, we have the expression valid for any sector  $Q$  and any type of excitations:

$$\frac{\Phi(P, QP^{-1})}{\Phi(P\Pi_{ii+1}, Q(P\Pi_{ii+1})^{-1})} = \alpha_{p_i p_{i+1}} \equiv \frac{ia_{p_i} - ia_{p_{i+1}} + \frac{u}{2}}{ia_{p_i} - ia_{p_{i+1}} - \frac{u}{2}} \quad \text{for } i = K - k, \dots, K - 1. \quad (3.71)$$

Before applying the whole Hamiltonian (3.36), which also represents the periodic boundary conditions, on the coefficients  $\Phi(P, QP^{-1})$ , we rewrite the obtained equation (3.71) in a form gathering all possibilities of excitations  $\bar{A} = (A_1, \dots, A_K)$ . At first we introduce an object (henceforth we will rename  $QP^{-1}$  simply by  $Q$ )

$$\hat{\Phi}(P) \equiv \sum_{Q, \bar{A}} \Phi(P, Q, \bar{A}) |A_{q_1}, \dots, A_{q_K}\rangle \quad (3.72)$$

where the sum is over all types of excitations and all corresponding sectors. The vector  $|A_{q_1}, \dots, A_{q_K}\rangle$  belongs to  $V_1 \otimes \dots \otimes V_K$  (where  $V = \text{span}\{e^{2\uparrow}, e^{3\uparrow}\}$ ) and represents one combination of  $K$  excitations. To explain the notation we write an example for  $K = 2$  excitations. In this case we have 4 types of different states:

$$\begin{aligned} \sum_{Q, \bar{A}} \Phi(P, Q, \bar{A}) |A_{q_1}, A_{q_2}\rangle &= \Phi(P, id, 1) |2\uparrow, 2\uparrow\rangle + \Phi(P, id, 2) |2\uparrow, 3\uparrow\rangle + \\ &\quad \Phi(P, \Pi_{12}, 2) |3\uparrow, 2\uparrow\rangle + \Phi(P, id, 3) |3\uparrow, 3\uparrow\rangle. \end{aligned}$$

Thus, for  $K = 2$  excitations we can express (3.71) by

$$\begin{aligned} \hat{\Phi}(\Pi_{12}P) &= \alpha_{12}^{-1} \left( \Phi(P, id, 1) |2\uparrow, 2\uparrow\rangle + \Phi(P, id, 2) |3\uparrow, 2\uparrow\rangle + \right. \\ &\quad \left. \Phi(P, \Pi_{12}, 2) |2\uparrow, 3\uparrow\rangle + \Phi(P, id, 3) |3\uparrow, 3\uparrow\rangle \right) \equiv S_{12}^{(2)} \hat{\Phi}(P) \end{aligned}$$

where we introduced, as in the first section, a new “ $S$ -matrix”, noted  $S_{12}^{(2)}$ . One can see that this  $S$ -matrix in this case is a pure permutation.

The same reasoning could be repeated for an arbitrary number  $K$  of excitations and we obtain

$$\hat{\Phi}(\Pi_{ab}P) = \alpha_{ab}^{-1} P_{ab} \hat{\Phi}(P), \quad (3.73)$$

where  $a = p(i)$ ,  $b = p(i+1)$  and the permutation  $P_{ab}$  acts nontrivially only on  $V_a \otimes V_b$  vector spaces. Obviously, the matrix  $S_{12}^{(2)}$  being a permutation, satisfies the Yang-Baxter equation.

To obtain the Bethe equations we apply the periodic boundary conditions which are automatically implied if we pass to the initial equation (3.36). We choose  $j = N$ , to get

$$\Psi_Q^{(N-1)}(\mathbf{x}) = \Lambda_N \Psi_Q(\mathbf{x}).$$

If no  $x$  equals  $N$ , we find the eigenvalue  $\Lambda_N = \sigma_N(a_1) \dots \sigma_N(a_K)$  where  $\sigma$  is the eigenvalue of one excitation function (3.50). Otherwise, the equation

$$\Psi_Q^{(N-1)}(x_1, \dots, N, \dots, x_K) = \Lambda_N \Psi_Q(x_1, \dots, N, \dots, x_K)$$

leads to additional condition on the coefficients  $\Phi(P, Q)$ . Hence, in terms of  $\hat{\Phi}(P)$ , we have

$$\hat{\Phi}(PC) = \prod_{k=1}^N \sigma_k(a_{p_1}) \hat{\Phi}(P)$$

where  $C$  is a cyclic permutation given by  $C = \Pi_{K1} \dots \Pi_{KK-1}$ .

Choosing  $P = C^{K-m}$ , we can derive a system of equations on coefficients  $\hat{\Phi}(id)$  which will be called “auxiliary problem 2”:

$$S_{m+1m}^{(2)} \dots S_{Km}^{(2)} S_{1m}^{(2)} \dots S_{m-1m}^{(2)} \hat{\Phi}(id) = \prod_{k=1}^N \sigma_k(a_m) \hat{\Phi}(id) \quad \text{for } m = 1, \dots, K. \quad (3.74)$$

At this stage, we arrived to the third step of nested coordinate Bethe ansatz. We have only 2 types of possible excitations  $e^{2\uparrow}$  and  $e^{3\uparrow}$  and the  $S$ -matrix (3.73) is a pure permutation.

### 3.5 Permutation problem, level three

As mentioned above, we have only 2 types of “particles”:  $e^{2\uparrow}$  and  $e^{3\uparrow}$  and an equation composed only with permutations:

$$\mathcal{H}_j \phi = P_{j+1j} \dots P_{Kj} P_{1j} \dots P_{j-1j} \phi = \Lambda \phi. \quad (3.75)$$

Again, we solve the problem with the coordinate Bethe ansatz, and fix the first “particle”  $e^{2\uparrow}$  as the “new” vacuum state

$$\phi_{M=0} = \prod_{i=1}^K e_i^{2\uparrow}, \quad \Lambda = 1. \quad (3.76)$$

The second “particle” ( $e^{3\uparrow}$ ) represents some excitations above this vacuum state. One can verify that the function

$$\phi_M(\vec{n}) = \sum_{x_1 < x_2 < \dots < x_M} \Psi(\vec{x}) \prod_{i=1}^M e_{x_i}^{3\uparrow} \quad (3.77)$$

with

$$\Psi(\vec{x}) = \sum_{P \in S_M} \Phi(P) \prod_{i=1}^M g_{x_i}(a_{p(i)}), \quad g_x(a) = a^x \quad (3.78)$$

is an eigenfunction of this permutation problem if the coefficients  $\Phi(P)$  obey some conditions that we formulate below.

Acting with the Hamiltonian  $\mathcal{H}_j$  gives only the periodicity condition on the coefficients  $\Phi(P)$ :

$$\frac{\Phi(PC)}{\Phi(P)} = [a_{p(M)}]^K, \quad C = \Pi_{1M} \dots \Pi_{M-1M}. \quad (3.79)$$

We assume a simple condition on the coefficients  $\Phi(P)$ :

$$\Phi(P\Pi_{ii+1}) = \Phi(P) \quad \text{for } i = 1, \dots, M. \quad (3.80)$$

Imposing these conditions and periodicity leads to

$$g_x(a(n)) = e^{\frac{2\pi i}{K}nx}, \quad n = 1, \dots, K. \quad (3.81)$$

Therefore the eigenvector can be written in the form

$$\phi_M(\vec{n}) = \sum_{x_1 < x_2 < \dots < x_M} \sum_{P \in S_M} \prod_{i=1}^M e^{\frac{2\pi i}{K}n_{p(i)}x_i} e^{3\uparrow_{x_i}} \quad \text{with } 1 \leq n_1 < n_2 < \dots < n_M \leq K. \quad (3.82)$$

It has the eigenvalue

$$\Lambda(\vec{n}) = \prod_{j=1}^M e^{\frac{2\pi i}{K}n_j} = \exp\left(\frac{2\pi i}{K}|\vec{n}|\right). \quad (3.83)$$

Then, the total number of states  $\phi_M(n_1, \dots, n_M)$  is

$$\sum_{0 \leq n_1 < n_2 < \dots < n_M \leq K-1} 1 = \binom{K}{M} = \frac{K!}{M!(K-M)!} \quad (3.84)$$

which is the right number of eigenfunctions with  $M$  excitations. Hence the conditions (3.80) and the functions (3.81) solve the permutation problem (3.75).

Gathering the results of sections 3.3, 3.4 and 3.5, we get the Bethe equations written in section 3.2.

#### 4 Generalization to $gl(\mathbf{n}|\mathbf{m}) \oplus gl(2)$ model

In this section we generalize the results obtained in previous sections to larger algebras, namely the case of  $gl(\mathbf{n}|\mathbf{m})_{\uparrow} \oplus gl(2)_{\downarrow}$  Hubbard model. The Hamiltonian is given by the expression

$$H_{gl(\mathbf{n}|\mathbf{m}) \oplus gl(2)} = \sum_{x=1}^L [(\Sigma P)_{\uparrow x, x+1} + (\Sigma P)_{\downarrow x, x+1}] + u \sum_{x=1}^L (C_{\uparrow x} C_{\downarrow x}), \quad (4.1)$$

with the notation:

$$(\Sigma P)_{\uparrow x, x+1} = \sum_{a=2}^{\mathbf{n}+\mathbf{m}} \left( E_{\uparrow x}^{a1} E_{\uparrow x+1}^{1a} + (-1)^{[a]} E_{\uparrow x}^{1a} E_{\uparrow x+1}^{a1} \right) \quad (4.2)$$

$$(\Sigma P)_{\downarrow x, x+1} = E_{\downarrow x}^{12} E_{\downarrow x+1}^{21} + E_{\downarrow x}^{21} E_{\downarrow x+1}^{12} \quad (4.3)$$

$$C_{\uparrow x} = E_{\uparrow x}^{11} - \sum_{a=2}^{\mathbf{n}+\mathbf{m}} E_{\uparrow x}^{aa}; \quad C_{\downarrow x} = E_{\downarrow x}^{11} - E_{\downarrow x}^{22} \quad (4.4)$$

and the grading we use is given in (2.1).

### 4.1 Result for $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$

We first gather all results detailed in the following sections and write down the Bethe equations of our model  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$ :

$$e^{ik_j L} = (-1)^{K+N+1} \prod_{m=1}^K \frac{i \sin k_j + ia_m + \frac{u}{4}}{i \sin k_j + ia_m - \frac{u}{4}}, \quad j = 1, \dots, N \quad (4.5)$$

$$(-1)^N \prod_{j=1}^N \frac{i \sin k_j + ia_m + \frac{u}{4}}{i \sin k_j + ia_m - \frac{u}{4}} = \Lambda(\vec{n}^{(3)}) \prod_{l=1, l \neq m}^K \frac{ia_m - ia_l + \frac{u}{2}}{ia_m - ia_l - \frac{u}{2}}, \quad m = 1, \dots, K \quad (4.6)$$

$$\Lambda(\vec{n}^{(3)}) = \exp\left(\frac{2i\pi}{K} |\vec{n}^{(3)}|\right), \quad |\vec{n}^{(3)}| = \sum_{i=1}^M n_i^{(3)}$$

$$0 \leq M \leq K \quad \text{and} \quad 1 \leq n_1^{(3)} < n_2^{(3)} < \dots < n_M^{(3)} \leq K \quad (4.7)$$

where  $L$  is the number of sites considered in Hubbard model,  $N$  is the total number of  $e^{2\downarrow}, e^{2\uparrow}, e^{3\uparrow}, \dots, e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$  “particles”.  $K$  counts the total number of excitations from  $e^{2\uparrow}$  to  $e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$  and finally  $M$  numbers the  $e^{3\uparrow}, \dots, e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$  “particles”.

There are Bethe parameters  $n_i^{(k)}$ ,  $3 < k \leq \mathfrak{m} + \mathfrak{n}$ , for each particle  $e^{k\uparrow}$ , but they don't show up in the Bethe equations. In section 4.4, it is shown more precisely how all these remaining parameters (that are quantized) appear in the Bethe ansatz construction.

The energies associated to these states are given by

$$E = (L - 2N) + 2 \sum_{m=1}^N \cos k_m \quad (4.8)$$

and their momentum reads

$$\mathfrak{p} = \sum_{m=1}^N k_m. \quad (4.9)$$

Let us note that the Bethe equations for  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$  are very close to the ones obtained for  $gl(2|1) \oplus gl(2)$ . This is due to the particular projectors we have chosen, see eq. (4.4). More general models (and Bethe equations) can be obtained varying these projectors: we come back on this point in section 5.

### 4.2 Coordinate Bethe ansatz, level 1

We solve this model via the coordinate Bethe ansatz. In this model we have  $\mathfrak{n} + \mathfrak{m} + 2$  types of different “particles” denoted by  $e^{1\uparrow}, e^{2\uparrow}, \dots, e^{(\mathfrak{n}+\mathfrak{m})\uparrow}$  and  $e^{1\downarrow}, e^{2\downarrow}$ . As in previous case we choose the vacuum as

$$\phi_0 = \prod_{x=1}^L e_x^{1\uparrow} e_x^{1\downarrow}.$$

The excitations above the vacuum state are given by

$$\phi[\bar{A}] = \sum_{\mathbf{x}} \Psi[\mathbf{x}, \bar{A}] e_{x_1}^{A_1} \dots e_{x_N}^{A_N} \quad (4.10)$$

with indices  $A_j = 2 \uparrow, 3 \uparrow, \dots, (\mathbf{n} + \mathbf{m}) \uparrow, 2 \downarrow$  corresponding to vectors  $e^{2\uparrow}, e^{3\uparrow}, \dots, e^{(\mathbf{n}+\mathbf{m})\uparrow}, e^{2\downarrow}$ . The sum over  $\mathbf{x}$  is again considered without points where two “particles” with the same spin are on the same site.  $N$  describes the number of all excitations and goes from 1 to  $L$ . In (4.10), the sites carrying vectors  $e_j^{1\uparrow} e_j^{1\downarrow}$ , not associated to any excitation, have been omitted.

Now we assume the *Bethe ansatz* for  $\Psi(\mathbf{x})$  and follow the steps detailed in section 3. We divide the coordinate space  $(x_1, x_2, \dots, x_N)$  into  $N!$  sectors. For  $x_{q_1} < x_{q_2} < \dots < x_{q_N}$ , we have

$$\Psi_Q(\mathbf{x}) = \sum_P (-1)^{[sg(Q)]} \Phi(P, QP^{-1}) e^{iP\mathbf{k} \cdot Q\mathbf{x}} \quad (4.11)$$

where  $P = [p_1, p_2, \dots, p_N]$  and  $Q = [q_1, q_2, \dots, q_N]$  are two permutations of the integers  $1, 2, \dots, N$  and  $P\mathbf{k} \cdot Q\mathbf{x} = \sum_i k_{p_i} x_{q_i}$ . As in previous section, the symbol  $(-1)^{[sg(Q)]}$  stands for the signature of the  $Q$ -permutation when restricted to fermionic particles  $e^{(\mathbf{n}+1)\uparrow}, \dots, e^{(\mathbf{n}+\mathbf{m})\uparrow}$ . We recall that  $\Psi(\mathbf{x})$  and accordingly  $\Phi(P, QP^{-1})$  both depend on the type of excitations  $\bar{A}$ .

We gather all the coefficients  $\Phi[P, QP^{-1}, \bar{A}]$  in a vector

$$\hat{\Phi}(P) \equiv \sum_{Q, \bar{A}} \Phi[P, QP^{-1}, \bar{A}] |A_{q_1}, \dots, A_{q_N} \rangle$$

where the sum is done over all possible types  $\bar{A}$  and all corresponding sectors  $Q$ . A vector  $|A_{q_1}, \dots, A_{q_N} \rangle$  represents one state of  $N$  possible excitations and belongs to  $V_1 \otimes \dots \otimes V_N$  (where  $V = \text{span}\{2 \downarrow; 3 \uparrow; 4 \uparrow; \dots; (\mathbf{n} + \mathbf{m}) \uparrow\}$ ).

In the case of only two particles ( $N = 2$ ) we are able to introduce the  $S$ -matrix:

$$\hat{\Phi}(\Pi_{12}P) = S_{12}^{(1)}(\lambda_1 - \lambda_2) \hat{\Phi}(P).$$

Here,  $S_{12}^{(1)}(\lambda_1 - \lambda_2) \equiv S_{12}^{(1)}$  acts on elementary vectors  $|A_1, A_2 \rangle$  as

$$S_{12}^{(1)} |2 \downarrow, 2 \downarrow \rangle = -|2 \downarrow, 2 \downarrow \rangle, \quad (4.12)$$

$$S_{12}^{(1)} |2 \downarrow, A \rangle = t_{12} |2 \downarrow, A \rangle + r_{12} |A, 2 \downarrow \rangle, \quad (4.13)$$

$$S_{12}^{(1)} |A, B \rangle = -|B, A \rangle, \quad \text{for } A, B = 2 \uparrow, \dots, (\mathbf{n} + \mathbf{m}) \uparrow \quad (4.14)$$

where

$$t_{12} = \frac{2i(\lambda_1 - \lambda_2)}{u + 2i(\lambda_1 - \lambda_2)}, \quad r_{12} = \frac{-u}{u + 2i(\lambda_1 - \lambda_2)}.$$

For an arbitrary number of excitation  $N$  we have

$$\hat{\Phi}(\Pi_{ab}P) = S_{ab}^{(1)}(\lambda_a - \lambda_b) \hat{\Phi}(P),$$

where the matrix  $S_{ab}^{(1)}$  acts nontrivially only on  $V_a \otimes V_b$  vector spaces.

Again, the periodic boundary condition on the function  $\Psi(\mathbf{x})$

$$\Psi_{QC}(\mathbf{x} + \mathbf{e}_{q_1}L) = \Psi_Q(\mathbf{x}) \quad \text{with } C = \Pi_{N1} \dots \Pi_{NN-1}$$

leads to the second step in our problem:

$$S_{j+1j}^{(1)} \dots S_{Nj}^{(1)} S_{1j}^{(1)} \dots S_{j-1j}^{(1)} \hat{\Phi}(id) = e^{ik_j L} \hat{\Phi}(id) \quad j = 1, \dots, N. \quad (4.15)$$

### 4.3 Auxiliary problem, level 2

As in the previous section, we transform slightly the  $S$ -matrix  $S_{12}^{(1)} \rightarrow -S_{12}^{(1)} \equiv S_{12}$  to simplify the calculations. The idea is that we work in different steps to diagonalize these matrices  $S_{12}$  via the coordinate Bethe ansatz. At each step we specify a “new” vacuum and “new” excitations. At this step we have  $\mathbf{n} + \mathbf{m}$  types of different excitations:  $e^{2\downarrow}, e^{2\uparrow}, e^{3\uparrow}, \dots, e^{(\mathbf{n}+\mathbf{m})\uparrow}$ .

Choosing the “new” vacuum as

$$\phi_0 = \prod_{k=1}^N e_k^{2\downarrow}, \quad (4.16)$$

the state of  $K$  excitations of any type is written as

$$\phi(\bar{A}) = \sum_{\mathbf{x}} \Psi(\mathbf{x}, \bar{A}) e_{x_1}^{A_1} \dots e_{x_K}^{A_K} \quad (4.17)$$

where the sum is done over all coordinates  $x_i$  without coinciding points  $x_i = x_k$  for any  $i, k$  (exclusion principle).  $\bar{A}$  is a vector  $(A_1, \dots, A_K)$  with  $A_i = 2 \uparrow, 3 \uparrow, \dots, (\mathbf{n} + \mathbf{m}) \uparrow$  (corresponding to  $e^{2\uparrow}, e^{3\uparrow} \dots e^{(\mathbf{n}+\mathbf{m})\uparrow}$ ). Again, sites carrying  $e^{2\downarrow}$  (no excitation) have been omitted in (4.17).

The Bethe ansatz for the coefficients  $\Psi(\mathbf{x}, \bar{A})$ , in the sector  $x_{q_1} < x_{q_2} < \dots < x_{q_K}$  where  $Q = [q_1, \dots, q_K]$  is the permutation of the integers  $1, 2, \dots, K$ , is given by

$$\Psi_Q(\mathbf{x}, \bar{A}) = \sum_P \Phi(P, QP^{-1}, \bar{A}) f_{x_{q_1}}(a_{p_1}) f_{x_{q_2}}(a_{p_2}) \dots f_{x_{q_K}}(a_{p_K}). \quad (4.18)$$

$f_x(a)$  is the one-particle solution (for any type of excitation  $A$ ) with rapidity  $a$ :

$$f_x(a) = \prod_{m=1}^{x-1} \left( -\frac{i\lambda_m + ia + \frac{u}{4}}{i\lambda_{m+1} + ia - \frac{u}{4}} \right). \quad (4.19)$$

The eigenvalue corresponding to this state  $\phi(\bar{A})$  takes the form

$$\Lambda_j = \sigma_j(a_1) \dots \sigma_j(a_K) \quad (4.20)$$

where  $\sigma_j(a)$  is the eigenvalue of the one-particle solution

$$\sigma_j(a) = -\frac{i\lambda_j + ia + \frac{u}{4}}{i\lambda_j + ia - \frac{u}{4}}. \quad (4.21)$$

Next we gather all the coefficients  $\Phi(P, QP^{-1}, \bar{A})$  in a vector

$$\hat{\Phi}(P) \equiv \sum_{Q, \bar{A}} \Phi(P, QP^{-1}, \bar{A}) |A_{q_1}, \dots, A_{q_K} \rangle \quad (4.22)$$

where the sum is over all types of excitations and all corresponding sectors. The vector  $|A_{q_1}, \dots, A_{q_K} \rangle$  belongs to  $V_1 \otimes \dots \otimes V_K$  (where  $V = \text{span}\{e^{2\uparrow}; e^{3\uparrow}; \dots; e^{(\mathbf{n}+\mathbf{m})\uparrow}\}$ ) and represents one combination of  $K$  excitations.

Application of the Hamiltonian (4.15) on the excited state  $\phi(\bar{A})$  gives two types of conditions imposed on the coefficients  $\hat{\Phi}(P)$ . From the first one, in the case of only two “particles” ( $K = 2$ ), we can introduce again a  $S$ -matrix corresponding to the second step. The second type implies the periodicity condition. Thus, the  $S$ -matrix is defined by

$$\hat{\Phi}(\Pi_{12}P) = S_{12}^{(2)}(a_1 - a_2)\hat{\Phi}(P) \quad (4.23)$$

with

$$S_{12}^{(2)}(a_1 - a_2) = \alpha_{12}P_{12}, \quad \text{and} \quad \alpha_{12} = \frac{ia_1 - ia_2 + \frac{u}{2}}{ia_1 - ia_2 - \frac{u}{2}}. \quad (4.24)$$

For an arbitrary number  $K$  of excitations we have

$$\hat{\Phi}(\Pi_{ij}P) = S_{ij}^{(2)}(a_i - a_j)\hat{\Phi}(P), \quad (4.25)$$

where  $p^{-1}(j) - p^{-1}(i) = 1$  and the permutation  $S_{ij}^{(2)}(a_i - a_j)$  acts nontrivially only on  $V_i \otimes V_j$  vector spaces. The matrix  $S_{12}^{(2)}(a_1 - a_2)$  satisfies the Yang-Baxter equation since it is a permutation.

The periodic boundary conditions on  $\hat{\Phi}(P)$  implied by the action of the Hamiltonian (4.15) is written in following form:

$$S_{m+1m}^{(2)} \cdots S_{Km}^{(2)} S_{1m}^{(2)} \cdots S_{m-1m}^{(2)} \hat{\Phi}(id) = \prod_{k=1}^N \sigma_k(a_m) \hat{\Phi}(id) \quad \text{for} \quad m = 1, \dots, K, \quad (4.26)$$

where the  $S$ -matrix arguments were omitted for simplicity.

#### 4.4 Permutation problem, level 3

Thus, we arrive to the third step of nested coordinate Bethe ansatz. Here, we have  $e^{2\uparrow}$ ,  $e^{3\uparrow}$ ,  $\dots$ ,  $e^{(n+m)\uparrow}$  “particles” that move ‘freely’, the Hamiltonian  $\Gamma$  being constructed on permutations only:

$$\Gamma \phi = P_{j+1j} \cdots P_{Kj} P_{1j} \cdots P_{j-1j} \phi = \Lambda \phi. \quad (4.27)$$

Note that  $\Gamma$  is a cyclic permutation, and is independent from  $j$ .

We choose the “particle”  $e^{2\uparrow}$  as the vacuum state:

$$\phi_{M=0} = \prod_{i=1}^K e_i^{2\uparrow}, \quad \Lambda = 1, \quad (4.28)$$

and introduce the function

$$\phi_M^{(3)}(\bar{A}) = \sum_{\vec{x}} \Psi(\vec{x}) \prod_{i=1}^M e_{x_i}^{A_i}, \quad A_i = 3\uparrow, 4\uparrow, \dots, (n+m)\uparrow. \quad (4.29)$$

It describes a state with  $M$  excitations above the vacuum state  $\phi_{M=0}$ .

The coefficients  $\Psi(\vec{x})$  are defined in the sector  $1 \leq x_{q(1)} < x_{q(2)} < \dots < x_{q(M)} \leq K$ , with  $Q \in \mathfrak{S}_M$ , by

$$\Psi_Q(\vec{x}) = \sum_{P \in \mathfrak{S}_M} \Phi^{(3)}(P, QP^{-1}) \prod_{i=1}^M g_{x_{q(i)}}(a_{p(i)}^{(3)}), \quad g_x(a) = a^x. \quad (4.30)$$

One can verify that  $\phi_M^{(3)}(\vec{A})$  is an eigenfunction with the following eigenvalue

$$\Lambda = \prod_{i=1}^M a_i^{(3)} \tag{4.31}$$

if some conditions, which we precise below, are satisfied.

Application of the Hamiltonian gives only the periodicity condition on the coefficients  $\Phi^{(3)}(P, QP^{-1})$ :

$$\frac{\Phi^{(3)}(PC, QP^{-1})}{\Phi^{(3)}(P, QP^{-1})} = [a_{p(M)}^{(3)}]^K, \quad C = \Pi_{1M} \dots \Pi_{M-1M}. \tag{4.32}$$

As in section 3.5, we assume some relations on the coefficients  $\Phi^{(3)}(P, QP^{-1})$ , but, now, the form of these relations depend on whether the particles are identical or not. If, for a given  $i$ ,  $x_i$  and  $x_{i+1}$  correspond to identical particles we impose

$$\Phi^{(3)}(\Pi_{p(i)p(i+1)}P, QP^{-1}) = \Phi^{(3)}(P, QP^{-1}), \tag{4.33}$$

while, otherwise, we set

$$\Phi^{(3)}(\Pi_{p(i)p(i+1)}P, QP^{-1}\Pi_{p(i)p(i+1)}) = \Phi^{(3)}(P, QP^{-1}). \tag{4.34}$$

As we can see, there is a sector changing in the relations above, and we proceed recursively using the same methods as above. We introduce

$$\hat{\Phi}^{(3)}(P) \equiv \sum_{Q, \vec{A}} \Phi^{(3)}[P, Q, \vec{A}] |A_{q_1}, \dots, A_{q_M} \rangle \tag{4.35}$$

where the sum is over possible types  $\vec{A}$  and all corresponding sectors  $Q \in \mathfrak{S}_M$ . The vector  $|A_{q_1}, \dots, A_{q_N} \rangle$  represents one state with  $M$  excitations and belongs to  $V_1 \otimes \dots \otimes V_N$  (where  $V = \text{span}\{e^{3\uparrow}, e^{4\uparrow}, \dots, e^{(n+m)\uparrow}\}$ ). Then, relations (4.33) and (4.34) can be rewritten in the following form

$$\hat{\Phi}^{(3)}(\Pi_{ab}P) = S_{ab}^{(3)} \hat{\Phi}^{(3)}(P), \quad S_{ab}^{(3)} = P_{ab}. \tag{4.36}$$

Therefore, the periodic boundary conditions on  $\hat{\Phi}(P)$  implied by the action of the chain of permutations is written as

$$P_{m+1m} \dots P_{Mm} P_{1m} \dots P_{m-1m} \hat{\Phi}^{(3)}(id) = [a_m^{(3)}]^K \hat{\Phi}^{(3)}(id), \quad m = 1, \dots, M \tag{4.37}$$

but here we have already only  $e^{3\uparrow}, \dots, e^{(n+m)\uparrow}$  “particles” involved in the calculations. Thus, we arrive to the next level of nested coordinate Bethe ansatz, with, again, an Hamiltonian built on permutations only, and a new chain of length  $M$ .

Using the previous considerations, we repeat the same method and we “eliminate” one by one the “particles”  $e^{3\uparrow}, \dots$  up to  $e^{(n+m-1)\uparrow}$ , choosing it as the vacuum state at each nested level.

We suppose that we have  $M_3$  “particles” of type  $e^{3\uparrow}$ ,  $M_4$  of type  $e^{4\uparrow}, \dots, M_{n+m}$  of type  $e^{(n+m)\uparrow}$ , so that  $M_3 + M_4 + \dots + M_{n+m} = M$ . At each level  $k = 3, \dots, n + m - 1$ , we have



particles  $e^{(k+1)\uparrow}, \dots, e^{(n+m)\uparrow}$  as different types of excitations above the vacuum state built on  $e^{k\uparrow}$ . The eigenvector  $\hat{\Phi}^{(k)}(id)$  can be written in the same form as in (4.29) and (4.30), with the set of Bethe roots  $\{a_i^{(k+1)}\}_{i=1}^{M_{k+1}+\dots+M_{n+m}}$  and the coefficients  $\Phi^{(k+1)}(P, QP^{-1})$  with  $P, Q \in \mathfrak{S}_{M_{k+1}+\dots+M_{n+m}}$ . These coefficients are used to write the vector  $\hat{\Phi}^{(k+1)}(id)$  that obeys the periodicity condition:

$$P_{m+1,m} \cdots P_{M_{k+1}+\dots+M_{n+m},m} P_{1m} \cdots P_{m-1,m} \hat{\Phi}^{(k+1)}(id) = \left[ a_m^{(k+1)} \right]^{M_k+\dots+M_{n+m}} \hat{\Phi}^{(k+1)}(id), \quad (4.38)$$

for  $m = 1, \dots, (M_{k+1} + \dots + M_{n+m})$ .

Also, we find the periodicity condition of the previous level (when we pass from level  $k-1$  to level  $k$ )

$$\left[ a_m^{(k)} \right]^{M_{k-1}+\dots+M_{n+m}} = \prod_{i=1}^{M_{k+1}+\dots+M_{n+m}} a_i^{(k+1)} \quad (4.39)$$

for  $m = 1, \dots, (M_{k+1} + \dots + M_{n+m})$  and  $k = 3, \dots, n+m-1$  with  $M_2 = K - (M_3 + \dots + M_{n+m})$ .

At last level, we have only one type of excitations  $e^{(n+m)\uparrow}$  on the vacuum state  $e^{(n+m-1)\uparrow}$  and we can see that it is the same case as in the permutation problem of the model  $gl(2|1) \oplus gl(2)$ . Thus, using the relation (4.39) for  $k = n+m-1$ , we find the following Bethe equations which link the case of one type excitation  $e^{(n+m)\uparrow}$  and the previous level with two types of excitations  $\{e^{(n+m-1)\uparrow}, e^{(n+m)\uparrow}\}$ :

$$\left[ a_i^{(n+m-1)} \right]^{M_{n+m-2}+M_{n+m-1}+M_{n+m}} = \prod_{j=1}^{M_{n+m}} a_j^{(n+m)}, \quad i = 1, \dots, M_{n+m-1} + M_{n+m}, \quad (4.40)$$

together with the result obtained from  $gl(2|1) \oplus gl(2)$  model

$$a_j^{(n+m)} = e^{2\pi i \frac{n_j^{(n+m)}}{M_{n+m-1}+M_{n+m}}}, \quad j = 1, \dots, M_{n+m} \quad (4.41)$$

$$1 \leq n_1^{(n+m)} < n_2^{(n+m)} < \dots < n_{M_{n+m}}^{(n+m)} \leq M_{n+m-1} + M_{n+m}. \quad (4.42)$$

In the same way, we can write the Bethe equations corresponding to the transition between level with  $\{e^{(n+m-1)\uparrow}, e^{(n+m)\uparrow}\}$  and the previous one

$$\left[ a_i^{(n+m-2)} \right]^{M_{n+m-3}+\dots+M_{n+m}} = \prod_{j=1}^{M_{n+m-1}+M_{n+m}} a_j^{(n+m-1)}, \quad i = 1, \dots, M_{n+m-2} + \dots + M_{n+m} \quad (4.43)$$

and we can continue this recurrence up to  $a^{(3)}$ .

Solutions for every set of Bethe roots  $a^{(k)}$  can be computed as following. For  $k = 3, \dots, n+m-1$  with  $M_2 = K - (M_3 + \dots + M_{n+m})$  we have

$$a_j^{(k)} = e^{\frac{2\pi i}{M_{k-1}+\dots+M_{n+m}} \left[ n_j^{(k)} + \frac{\sum_{m=k+1}^{n+m} \sum_{i=1}^{M_m+\dots+M_{n+m}} n_i^{(m)}}{M_k+\dots+M_{n+m}} \right]}, \quad j = 1, \dots, (M_k + \dots + M_{n+m})$$

$$1 \leq n_1^{(k)} < n_2^{(k)} < \dots < n_{M_k+\dots+M_{n+m}}^{(k)} \leq M_{k-1} + \dots + M_{n+m}. \quad (4.44)$$

Therefore, the function is the eigenvector of the initial permutation problem (4.27)

$$\phi_M^{(3)}(\bar{A}) = \sum_{Q \in \mathfrak{S}_M} \sum_{\bar{x} \in Q} \sum_{P \in \mathfrak{S}_M} \Phi^{(3)}(P, QP^{-1}) \prod_{i=1}^M [a_{p(i)}^{(3)}]^{x_{q(i)}} e_{x_i}^{A_i} \phi_0(2 \uparrow) \quad (4.45)$$

where we denoted explicitly the ‘empty sites’ (with no excitation) as  $\phi_0(2 \uparrow)$ .

The coefficients  $\Phi^{(3)}(id, Q)$  gathered in  $\hat{\Phi}^{(3)}(id)$  are connected with the next level coefficients  $\Phi^{(4)}(id, Q)$ . To see it, we fix the value of  $M_3$  and consider vectors  $\bar{B}$  which characterizes excitations of the form

$$\bar{B} = \overbrace{((\mathbf{n} + \mathbf{m}) \uparrow, \dots, (\mathbf{n} + \mathbf{m}) \uparrow, \dots, 4 \uparrow, \dots, 4 \uparrow)}^{M-M_3} \overbrace{(3 \uparrow, \dots, 3 \uparrow)}^{M_3}. \quad (4.46)$$

Then, we consider the restriction of the relation (4.35) to

$$\hat{\Phi}^{(3)}(id) \Big|_{\text{restrict.}} = \sum_{Q, \bar{B}} \Phi^{(3)}[id, Q, \bar{B}] |B_{q_1}, \dots, B_{q_M} \rangle. \quad (4.47)$$

One can recognize in this term the  $\phi_{M-M_3}^{(4)}(\bar{B})$  coefficient (here we take only first  $M - M_3$  values  $\bar{B}$ ):

$$\sum_Q \Phi^{(3)}[id, Q, \bar{B}] e_{q^{-1}(1)}^{B_1} \dots e_{q^{-1}(M-M_3)}^{B_{M-M_3}} \phi_0(3 \uparrow) = \sum_{\vec{y} \in [1, M]} \Psi(\vec{y}) \prod_{i=1}^{M-M_3} e_{y_i}^{B_i} \phi_0(3 \uparrow) \equiv \phi_{M-M_3}^{(4)}(\bar{B}), \quad (4.48)$$

with  $y_i = q^{-1}(i)$  for  $i = 1, \dots, M - M_3$ . In the left hand side of the equation  $e_{q^{-1}(i)}^{B_i}$  are the operators which create the corresponding excitations  $B_i$  on the site  $q^{-1}(i)$  of the chain of particles  $e^{3 \uparrow}$ .

Therefore, using the same ansatz as in (4.30) for  $\Psi(\vec{y})$  in  $\phi_{M-M_3}^{(4)}(\bar{B})$  we can identify the coefficients  $\Phi^{(3)}(id, Q)$  as

$$\Phi^{(3)}(id, Q, \bar{B}) = \sum_{P \in \mathfrak{S}_{M-M_3}} \Phi^{(4)}(P, Q'P^{-1}, \bar{B}) \prod_{i=1}^{M-M_3} [a_{p(i)}^{(4)}]^i, \quad (4.49)$$

with  $Q' \in \mathfrak{S}_{M-M_3}$  defined by  $q'(i) = q(i)$  for  $i = 1, \dots, M - M_3$  and

$$B_i = 4 \uparrow, \dots, (\mathbf{n} + \mathbf{m}) \uparrow, \quad i = 1, \dots, M - M_3. \quad (4.50)$$

In the general case, the coefficients  $\Phi^{(k)}(P, Q)$  are defined by the same relations: for  $k = 3, \dots, \mathbf{n} + \mathbf{m} - 2$ , we have

$$\Phi^{(k)}(id, Q, \bar{B}) = \sum_{P \in \mathfrak{S}_{M_{k+1} + \dots + M_{\mathbf{n} + \mathbf{m}}}} \Phi^{(k+1)}(P, Q'P^{-1}, \bar{B}) \prod_{i=1}^{M_{k+1} + \dots + M_{\mathbf{n} + \mathbf{m}}} [a_{p(i)}^{(k+1)}]^i, \quad (4.51)$$

where  $Q' \in \mathfrak{S}_{M_{k+1} + \dots + M_{\mathbf{n} + \mathbf{m}}}$  is defined by  $q'(i) = q(i)$  for  $i = 1, \dots, M_{k+1} + \dots + M_{\mathbf{n} + \mathbf{m}}$ ,

$$B_i = (k + 1) \uparrow, \dots, (\mathbf{n} + \mathbf{m}) \uparrow, \quad i = 1, \dots, M_{k+1} + \dots + M_{\mathbf{n} + \mathbf{m}}$$

and there are relations similar to (4.33) and (4.34).

At last, when  $k = n + m - 1$ , using the results of  $gl(2|1) \oplus gl(2)$  model, we get

$$\Phi^{(n+m-1)}(id, Q, \bar{B}) = \sum_{P \in \mathfrak{S}_{M_{n+m-1} + M_{n+m}}} \prod_{i=1}^{M_{n+m-1} + M_{n+m}} [a_{p(i)}^{(n+m)}]^i. \quad (4.52)$$

Equations of the type (4.49) and (4.51) together with relations (4.33) and (4.34) allow us to derive all the coefficients  $\Phi^{(3)}(P, Q)$ . The eigenvalue reads

$$\Lambda(\vec{n}^{(3)}, \dots, \vec{n}^{(n+m)}) \equiv \Lambda(\vec{n}^{(3)}) = \prod_{i=1}^M a_i^{(3)} = \exp\left(\frac{2\pi i}{K} \sum_{i=1}^M n_i^{(3)}\right) = \exp\left(\frac{2\pi i}{K} |\vec{n}^{(3)}|\right). \quad (4.53)$$

The Bethe parameters  $\vec{n}^{(k)}$ ,  $k > 3$ , ensure the correct multiplicity of eigenfunctions. Indeed, the total number of states  $\phi_M(\vec{n}^{(3)}, \vec{n}^{(4)}, \dots, \vec{n}^{(n+m)})$  is

$$\sum_{\substack{\vec{n}^{(3)}, \vec{n}^{(4)}, \dots, \vec{n}^{(n+m)} \\ 1 \leq n_i^{(k)} < n_{i+1}^{(k)} \leq J_k}} 1 = \frac{K!}{M_3! \dots M_{n+m}! (K - M)!}, \quad \text{where } J_k = \sum_{\ell=3}^{m+n-k-1} M_\ell, \quad (4.54)$$

which shows that the ansatz is complete.

## 5 Another $gl(2|2) \oplus gl(2)$ model

### 5.1 Comparison with AdS/CFT models

It is known that the Hubbard model can be connected to the  $SU(2)$  subsector of the super-Yang-Mills (SYM) theory. The dilatation operator in this subsector can be identified with a Hamiltonian that is very close to the Hubbard one. Although not exact,<sup>2</sup> this correspondence has shed a new light on the integrability aspect of SYM models. In fact, one can introduce perturbatively a scattering matrix (obeying the Yang-Baxter equation) that differs from the Hubbard one by a phase (so-called ‘wrapping problem’). This phase is also seen in the Bethe equation of the model.

The models we have presented up to now possess the same property: they have Bethe equations that are very close to the Hubbard Bethe equations, but a phase. Unfortunately, this phase is built on ‘hidden’ Bethe parameters, but not on the impulsions of our ‘particles’ (which is the case of SYM Bethe equations). Hence, the correspondence is not immediate, but the present construction gives a way to introduce a phase in the equations. To strengthen the present approach, we show in this section an example of another  $gl(2|2) \oplus gl(2)$  model using different choices of the projectors  $\pi$  and  $\bar{\pi}$ . It will lead to Bethe equation with a phase that (partially) depends on the impulsions of the particles.

We recall that the Hamiltonian is given by

$$H_{gl(2|2) \oplus gl(2)} = \sum_{x=1}^L \left( (\Sigma P)_{\uparrow x, x+1} + (\Sigma P)_{\downarrow x, x+1} + u C_{\uparrow x} C_{\downarrow x} \right), \quad (5.1)$$

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<sup>2</sup>The correspondence breaks down at level 4 of perturbation theory [18].

where we choose the projectors  $\pi_\uparrow$  and  $\pi_\downarrow$  such that  $\mathcal{N}_\uparrow = \{1, 2\}$  and  $\overline{\mathcal{N}}_\uparrow = \{3, 4\}$ . Thus

$$(\Sigma P)_{\uparrow x, x+1} = \sum_{i=1}^2 \sum_{j=3}^4 \left( E_{\uparrow x}^{ij} E_{\uparrow x+1}^{ji} + E_{\uparrow x}^{ji} E_{\uparrow x+1}^{ij} \right), \quad (5.2)$$

$$(\Sigma P)_{\downarrow x, x+1} = E_{\downarrow x}^{12} E_{\downarrow x+1}^{21} + E_{\downarrow x}^{21} E_{\downarrow x+1}^{12}, \quad (5.3)$$

$$C_{\uparrow x} = E_{\uparrow x}^{11} + E_{\uparrow x}^{22} - E_{\uparrow x}^{33} - E_{\uparrow x}^{44} \quad ; \quad C_{\downarrow x} = E_{\downarrow x}^{11} - E_{\downarrow x}^{22}. \quad (5.4)$$

We use the same approach, i.e. the coordinate Bethe ansatz, to solve this model. Here we do not give all explicit details of calculation. The method is the same, however there are some modifications appearing when we pass from the initial problem to the first auxiliary problem and then to the second auxiliary problem. We briefly give the most important statements as well as the Bethe equations.

## 5.2 Bethe equations for $gl(2|2) \oplus gl(2)$

As we shall see in the next subsection for construction of the Bethe ansatz, in this model there are four different kinds of “particles” above the vacuum state:  $e^{2\uparrow}$  which is defined by projectors above as  $\pi$ -particle and  $e^{2\downarrow}, e^{3\uparrow}$  and  $e^{4\uparrow}$  which are the  $\bar{\pi}$ -particles. To define the Bethe equations, we first introduce

$$\mathbb{A} = \{a_1, a_2, \dots, a_{N_1}\} \quad \text{for some integers such that} \quad 1 \leq a_1 < a_2 < \dots < a_{N_1} \leq N. \quad (5.5)$$

Then, the Bethe equations can be written as

$$e^{ik_j(L-N_2-N_3)} = (-1)^{N_1} \quad \text{for} \quad j \in \mathbb{A} \quad (5.6)$$

$$e^{ik_j L} = (-1)^{N+1-(N_1+N_2+N_3)} \prod_{m=1}^{N_2+N_3} \frac{i \sin k_j + ib_m + \frac{u}{4}}{i \sin k_j + ib_m - \frac{u}{4}}, \quad \text{for} \quad j \in [1, N] \setminus \mathbb{A} \quad (5.7)$$

$$(-1)^{N-N_1} \prod_{\substack{j=1 \\ j \notin \mathbb{A}}}^N \frac{i \sin k_j + ib_m + \frac{u}{4}}{i \sin k_j + ib_m - \frac{u}{4}} = \Lambda(\vec{n}) \prod_{j \in \mathbb{A}} e^{-ik_j} \prod_{\substack{l=1 \\ l \neq m}}^{N_2+N_3} \frac{ib_m - ib_l + \frac{u}{2}}{ib_m - ib_l - \frac{u}{2}}, \quad (5.8)$$

for  $m = 1, \dots, N_2 + N_3$

$$\Lambda(\vec{n}) = \exp \left( \frac{2i\pi}{N_2 + N_3} \sum_{i=1}^{N_3} n_i \right), \quad 1 \leq n_1 < n_2 < \dots < n_{N_3} \leq N_2 + N_3 \quad (5.9)$$

where  $L$  is the number of sites considered in Hubbard model,  $N$  is total number of all  $e^{2\uparrow}$ ,  $e^{2\downarrow}, e^{3\uparrow}$  and  $e^{4\uparrow}$  “particles”.  $N_1$  counts  $e^{2\uparrow}$  excitations,  $N_2, N_3$  count respectively  $e^{3\uparrow}$  and  $e^{4\uparrow}$  particles. Remark that with respect to the Bethe equations computed in the previous sections, the phase  $\Lambda(\vec{n})$  has been changed to

$$\Lambda(\vec{n}) \quad \rightarrow \quad \Lambda(\vec{n}) \prod_{j \in \mathbb{A}} e^{-ik_j}$$

showing a (partial) dependence on the momenta of the particles. This “dressing” of the phase is similar to the one suggested in [20].

The energy associated to the state is given by

$$E = L - 2(N - N_1) + 2 \sum_{j \in [1, N] \setminus \mathbb{A}} \cos(k_j) \quad (5.10)$$

and the momentum reads

$$\mathbf{p} = \sum_{j=1}^N k_j. \quad (5.11)$$

The set of integers  $\mathbb{A}$  is related to the  $\pi$ -particles in the first auxiliary problem and the integers  $n_j$  correspond to the Bethe parameters of the last level, but their Bethe equations have already been solved: they just correspond to the quantization of these parameters. The parameters  $\{b_l\}_{l=1, \dots, N_2+N_3} \equiv \{a_{N_1+l}\}_{l=1, \dots, N_2+N_3}$  correspond to  $\bar{\pi}$ -particles in the first auxiliary problem, and do have Bethe equations, see eq. (5.8).

### 5.3 Calculation description for $gl(2|2) \oplus gl(2)$

In this paragraph we briefly describe some important points of the approach for this new model. At the first level of coordinate Bethe ansatz, together with the  $\bar{\pi}$ -particles (the “physics” of which we studied above), we include some  $\pi$ -particles. The states of  $N$  excitations can be written as

$$\phi[\bar{A}] = \sum_{\mathbf{x}} \Psi[\mathbf{x}, \bar{A}] e_{x_1}^{A_1} \dots e_{x_N}^{A_N} \quad (5.12)$$

with  $A = (2, \uparrow); (3, \uparrow); (4, \uparrow); (2, \downarrow)$ .

This modifies the ansatz for the wave function as, for  $x_{q_1} < x_{q_2} < \dots < x_{q_N}$

$$\Psi_Q^{P_{\pi\bar{\pi}}}(\mathbf{x}) = \sum_{P' = P_{\pi}P_{\bar{\pi}}} \Phi(\hat{P}Q, \hat{P}^{-1}) e^{i\hat{P}\mathbf{k}\cdot\mathbf{x}}, \quad \hat{P} = P_{\pi\bar{\pi}}P' \quad (5.13)$$

with the energy:

$$E^{P_{\pi\bar{\pi}}} = 2 \sum_{l \in \bar{\pi}} \cos(k_{P_{\pi\bar{\pi}}(l)}) + L - 2(N - N_1) \quad (5.14)$$

where  $Q \in \mathfrak{S}_N$ . We have to consider the permutation of Bethe roots  $k_j$  in some factorized form:  $P' = P_{\pi}P_{\bar{\pi}}$  where the terms permute only  $\pi$  and  $\bar{\pi}$  particles separately. In addition, we could also vary the value of energy by mixing the impulsions of all particles, adding a permutation  $P_{\pi\bar{\pi}}$  in the term  $e^{iP\mathbf{k}\cdot\mathbf{x}}$ , but it does not produce new (independent) eigenvectors.

Applying the Hamiltonian (5.1) on the vector (5.12) we find the relations between the coefficients  $\Phi(\hat{P}Q, \hat{P}^{-1})$ . Again we can gather all relations in a vector  $\hat{\Phi}(P)$

$$\hat{\Phi}(P') \equiv \sum_{Q', \bar{A}} \Phi_{\bar{A}}(P', Q') |A_{Q'(1)}, \dots, A_{Q'(N)} \rangle$$

where we have defined  $P' \equiv \hat{P}Q \in \mathfrak{S}_N$ ,  $Q' \equiv \hat{P}^{-1} \in \mathfrak{S}_N$  and the sum is over all types of excitations and all corresponding sectors. The vector  $|A_1, \dots, A_N \rangle$  belongs to  $V_1 \otimes \dots \otimes V_N$ , where  $V = \text{span}\{2 \uparrow, 3 \uparrow, 4 \uparrow, 2 \downarrow\}$  and represents one type of  $N$  excitations.

Thus, the relations between  $\Phi(\hat{P}Q, \hat{P}^{-1})$  can be expressed using  $S$ -matrix presentation

$$\hat{\Phi}(\Pi_{12}P) = S_{12}^{(1)}(\lambda_1 - \lambda_2) \hat{\Phi}(P)$$



tions  $e^{3\uparrow}, e^{4\uparrow}$  and  $e^{2\downarrow}$  can be written as, for  $x_{q(1)} < x_{q(2)} < \dots < x_{q(N_1+N_2+N_3)}$  and  $Q \in \mathfrak{S}_{N_1+N_2+N_3}$

$$\Psi_Q(\mathbf{x}, \bar{A}) = \sum_{P \in \mathcal{P}_f} \Phi_{P^{-1}}^{PQ} \prod_{i=1}^{N_1} h_{x_i}(a_n) \prod_{n=1}^{N_2+N_3} f_{x_n}(a_{p(N_1+n)}) \quad (5.20)$$

where  $\mathcal{P}_f$  is the set of permutations acting on  $\bar{\pi}$ -particles only, and  $f_x(a)$  is defined in (4.19).

The Bethe parameters  $a_1, \dots, a_{N_1}$  being the arguments of  $h_x(a)$  are already quantized on the small chain  $[1, N]$  and we choose them as:

$$1 \leq a_1 < \dots < a_{N_1} \leq N.$$

There are two different cases possible: 1) there exists a Bethe root  $a_\alpha = j$  for some  $\alpha \in [1, N_1]$ , with  $j$  being the index in (5.17) and 2) there is no such Bethe root. In the first case, functions (5.18) are eigenvectors of (5.17) with the eigenvalue

$$S_{j+1j}^{(1)} \dots S_{Nj}^{(1)} S_{1j}^{(1)} \dots S_{j-1j}^{(1)} \hat{\Phi}(id) = (-1)^{N-N_1+1} e^{ik_j(N_2+N_3)} \hat{\Phi}(id). \quad (5.21)$$

As we can see, the remaining parameters  $a_i$  with  $i \in [N_1 + 1, N_1 + N_2 + N_3]$  as well as the coefficients  $\Phi_{P^{-1}}^{PQ}$  are not constrained in this first case: their Bethe equations is obtained from the second case.

In the second case, when there is no Bethe root  $a_\alpha = j$  for any  $\alpha \in [1, N_1]$ , (5.17) implies the conditions on the coefficients  $\Phi_{P^{-1}}^{PQ}$

$$\frac{\Phi_{P^{-1}}^{PQ\Pi_{ii+1}}}{\Phi_{P^{-1}}^{PQ\Pi_{q(i)q(i+1)}}^{-1}} = \frac{ia_{p(q(i))} - ia_{p(q(i+1))} - \frac{u}{2}}{ia_{p(q(i))} - ia_{p(q(i+1))} + \frac{u}{2}}, \quad (5.22)$$

for all  $p(q(i)), p(q(i+1))$  in  $[N_1 + 1, N_1 + N_2 + N_3]$  and

$$\frac{\Phi_{P^{-1}}^{PQ\Pi_{ii+1}}}{\Phi_{P^{-1}}^{PQ}} = e^{-ik_{a_{p(q(i))}} \sigma_{a_{p(q(i))}}(a_{p(q(i+1))})} \quad (5.23)$$

for all  $p(q(i)) \in [1, N_1]$ ,  $p(q(i+1)) \in [N_1 + 1, N_1 + N_2 + N_3]$  and  $\sigma_i(a)$  is defined in (4.21).

The calculations for the eigenvalue of (5.18) give

$$S_{j+1j}^{(1)} \dots S_{Nj}^{(1)} S_{1j}^{(1)} \dots S_{j-1j}^{(1)} \hat{\Phi}(id) = (-1)^{N_1} \prod_{i=1}^{N_2+N_3} \sigma_j(a_{i+N_1}) \hat{\Phi}(id). \quad (5.24)$$

The periodic boundary condition on the coefficients  $\Phi_{P^{-1}}^{PQ}$  is

$$\frac{\Phi_{P^{-1}}^{PQC_{N_1+N_2+N_3}}}{\Phi_{P^{-1}}^{PQ}} = (-1)^{N_1} \prod_{l=1}^N \sigma_l(a_{p(q(N_1+N_2+N_3))}) \quad (5.25)$$

for  $Q$  such that  $q(N_1+N_2+N_3) \in [N_1+1, N_1+N_2+N_3]$ .  $C_{N_1+N_2+N_3}$  is a cyclic permutation,  $C_K = \Pi_K^1 \dots \Pi_K^{K-1}$ .

The difficulty in this model and of all models with  $\pi$ -particles is that starting from the auxiliary problem we can not mix the Bethe roots of different types of excitations. Indeed, in (5.20) the permutation acts only on  $\bar{\pi}$ -particles, while  $\hat{\Phi}(P)$  mixes *a priori* any kind of particle. In this case we define again the vector  $\hat{\Phi}(P)$ , but only for  $\bar{\pi}$ -particles as actually it was defined in (3.72):

$$\hat{\Phi}(P) \equiv \sum_{Q \in \mathfrak{S}_{N_2+N_3}, \bar{A}} \Phi_Q^P(\bar{A}) |A_{q_1}, \dots, A_{q_{(N_2+N_3)}} \rangle \quad (5.26)$$

where the sum is over all types of  $\bar{\pi}$  excitations and all corresponding sectors. The vector  $|A_{q_1}, \dots, A_{q_K} \rangle$  is in  $V_1 \otimes \dots \otimes V_K$  with  $V = \text{span}\{e^{3\uparrow}, e^{4\uparrow}\}$ .

Working with the periodicity condition (5.25), we take for instance  $Q = id$ , which ensures that the constraint  $q(N_1 + N_2 + N_3) \in [N_1 + 1, N_1 + N_2 + N_3]$  is satisfied. Thus, in the left hand side, in the coefficient  $\Phi_{P^{-1}}^{\prod_{p(N_1+N_2+N_3)}^{p(1)} \dots \prod_{p(N_1+N_2+N_3)}^{p(N_1+N_2+N_3-1)} P}$  we can identify which particles are permuted by  $\prod_{p(N_1+N_2+N_3)}^{p(k)}$  and we can apply corresponding relation (5.22) or (5.23). Due to the fact that  $P \in \mathcal{P}_f$  does not act on  $\pi$ -particles, we can use relation (5.23) to extract from the coefficient the action of the last  $N_1$  permutations  $\prod_{p(N_1+N_2+N_3)}^{p(j)}$ ,  $j = 1, \dots, N_1$ , and get:

$$\prod_{i=1}^{N_1} e^{-ik_{a_i}} \sigma_{a_i}(a_{p(N_1+N_2+N_3)}) \Phi_{P^{-1}}^{\prod_{p(N_1+N_2+N_3)}^{p(N_1+1)} \dots \prod_{p(N_1+N_2+N_3)}^{p(N_1+N_2+N_3-1)} P} = (-1)^{N_1} \prod_{l=1}^N \sigma_l(a_{p(N_1+N_2+N_3)}) \Phi_{P^{-1}}^P \quad (5.27)$$

We can rewrite this condition using  $\hat{\Phi}(P)$  notation. Choosing  $P$  as a power of the cyclic permutation acting only on  $\bar{\pi}$ -particles,

$$P \equiv C^{N_2+N_3-m} = \left( \prod_{N_1+N_2+N_3}^{N_1+1} \dots \prod_{N_1+N_2+N_3}^{N_1+N_2+N_3-1} \right)^{N_2+N_3-m}, \quad (5.28)$$

the calculation becomes equivalent to the one of section 3.4, and we obtain:

$$\prod_{i=1}^{N_1} e^{-ik_{a_i}} \sigma_{a_i}(a_m) \hat{\Phi}(C^{N_2+N_3-m+1}) = (-1)^{N_1} \prod_{l=1}^N \sigma_l(a_m) \hat{\Phi}(C^{N_2+N_3-m}). \quad (5.29)$$

Thus, for  $m = N_1 + 1, \dots, N_1 + N_2 + N_3$

$$\begin{aligned} & \prod_{i=1}^{N_1} e^{-ik_{a_i}} \sigma_{a_i}(a_m) \prod_{\substack{l=N_1+1 \\ l \neq m}}^{N_1+N_2+N_3} \left( \frac{ia_l - ia_m - \frac{u}{2}}{ia_l - ia_m + \frac{u}{2}} \right) (P_{m m+1} \dots P_{m N_1+N_2+N_3} P_{m 1} \dots P_{m m-1}) \hat{\Phi}(id) = \\ & = (-1)^{N_1} \prod_{l=1}^N \sigma_l(a_m) \hat{\Phi}(id) \end{aligned} \quad (5.30)$$

and finally we get

$$\left( e^{2\pi i \frac{\sum_{i=1}^{N_3} n_i}{N_2+N_3}} \prod_{i=1}^{N_1} e^{-ik_{a_i}} \prod_{\substack{l=N_1+1 \\ l \neq m}}^{N_1+N_2+N_3} \left( \frac{ia_l - ia_m - \frac{u}{2}}{ia_l - ia_m + \frac{u}{2}} \right) - (-1)^{N_1} \prod_{\substack{l=1 \\ l \notin A}}^N \sigma_l(a_m) \right) \hat{\Phi}(id) = 0, \quad (5.31)$$

$m = N_1 + 1, \dots, N_1 + N_2 + N_3$



where we have introduced the set  $\mathbb{A} = \{a_1, a_2, \dots, a_{N_1}\} \subset [1, N]$  and the Bethe parameters  $n_j$  that label the eigenfunctions of the cyclic permutation (as in section 4.4).

To stress the difference between the quantized parameters  $a_j$ ,  $j \leq N_1$ , (that are similar to the parameters  $n_j$ ), and the parameters  $a_j$ ,  $j > N_1$ , we denoted the latter  $b_j \equiv a_{j+N_1}$  in the Bethe equations written in section 5.2.

## 6 $gl(2|1) \oplus gl(2)$ and $gl(2|2) \oplus gl(2)$ Hamiltonians

In previous sections we considered examples of Hubbard model with algebra  $gl(2|1)_\uparrow \oplus gl(2)_\downarrow$  and its generalization to  $gl(\mathfrak{n}|\mathfrak{m})_\uparrow \oplus gl(2)_\downarrow$  model. However only examples with “small” algebras like  $gl(2|1)_\uparrow \oplus gl(2)_\downarrow$  or  $gl(2|2)_\uparrow \oplus gl(2)_\downarrow$  seem to find applications in physics. Performing different Jordan-Wigner transformations one can write the corresponding Hamiltonians in terms of creation and annihilation operators.

### 6.1 Jordan-Wigner transformation

We briefly recall some relations of Jordan-Wigner transformation [21] (for more detailed explanations, see e.g. [16]). The Jordan-Wigner transformation essentially consists in the construction of a mapping

$$E^{ij} \in gl(2^{p-1}|2^{p-1}) \leftrightarrow \{c^\dagger, c; d^\dagger, d; e^\dagger, e; \dots\},$$

where  $c, d, e, \dots$  are fermionic operators. To present this construction, it is convenient to introduce a matrix  $X$

$$X = \begin{pmatrix} 1 - n^c & c \\ c^\dagger & n^c \end{pmatrix}, \quad n^c = c^\dagger c. \quad (6.1)$$

Its entries  $X_{\mu\nu}$  ( $\mu, \nu = 1, 2$ ) have a natural grading given by  $[\mu] + [\nu]$  where  $[1] = 1$  and  $[2] = 0$ . The mapping is given by the relation

$$E^{ij} \leftrightarrow (-1)^s X_{\mu_1\nu_1}^{(1)} \otimes X_{\mu_2\nu_2}^{(2)} \otimes \dots \otimes X_{\mu_p\nu_p}^{(p)} \quad (6.2)$$

where to every value  $i$  and  $j \in [1, 2^p]$  is associated with an element  $\{\mu_1, \mu_2, \dots, \mu_p\}$  and  $\{\nu_1, \nu_2, \dots, \nu_p\}$  respectively with  $\mu_i, \nu_i = 1$  or  $2$ . Total grading is given by  $s = \sum_{i=2}^p [\mu_i] \left( \sum_{j=1}^{i-1} ([\mu_j] + [\nu_j]) \right)$ .

Transformation for  $gl(2)_\downarrow$  algebra is simply given by identification of matrices  $E^{ij}$  and  $X_{\mu\nu}$ . But since  $gl(2)$  algebra contain only bosonic operators, in order to satisfy the anticommutation relation between fermionic operators on different sites ( $\{c_a^\dagger, c_b\} = 0$ ), one should introduce some factor to  $E_a^{ij}$  (elementary matrix  $E^{ij}$  on the site  $a$ ). To present the result, we gather the different matrices  $E^{ij}$ ,  $i, j = 1, 2$  into a formal matrix  $\mathbb{E}$ . Then, the Jordan-Wigner transformation for  $gl(2)$  algebra reads

$$\mathbb{E}_a = \begin{pmatrix} E_a^{11} & E_a^{12} \\ E_a^{21} & E_a^{22} \end{pmatrix} \equiv \begin{pmatrix} 1 - n_{a\downarrow}^c & c_{a\downarrow} Z_{a+1\downarrow}^L \\ c_{a\downarrow}^\dagger Z_{a+1\uparrow}^L & n_{a\downarrow}^c \end{pmatrix} \quad \text{with} \quad Z_{a+1\downarrow}^L = \prod_{x=a+1}^L (1 - 2n_{x\downarrow}^c) \quad (6.3)$$

Now consider an example for the cases of  $gl(2|2)_\uparrow$  and  $gl(2|1)_\uparrow$  algebras. For both algebra we perform one mapping and in the case of smaller algebra ( $gl(2|1)_\uparrow$ ) we remove

a subspace. We take the mapping  $\{1 \rightarrow 11, 2 \rightarrow 22, 3 \rightarrow 12, 4 \rightarrow 21\}$  as an example. In the initial problem, elements of  $gl(2|2)_\uparrow \oplus gl(2)_\downarrow$  and  $gl(2|1)_\uparrow \oplus gl(2)_\downarrow$  algebras transform into fermionic operators which are considered to anticommute even for different spins, e.g.  $\{c_\uparrow^\dagger, c_\uparrow\} = 0$ . This implies that the total Jordan-Wigner transformation is written as

$$\mathbb{E}_{a\uparrow} \equiv \begin{pmatrix} (1 - n_{a\uparrow}^c)(1 - n_{a\uparrow}^d) & -c_{a\uparrow}d_{a\uparrow} & (1 - n_{a\uparrow}^c)d_{a\uparrow}Z_{1\downarrow}^L & -c_{a\uparrow}(1 - n_{a\uparrow}^d)Z_{1\downarrow}^L \\ c_{a\uparrow}^\dagger d_{a\uparrow}^\dagger & n_{a\uparrow}^c n_{a\uparrow}^d & c_{a\uparrow}^\dagger n_{a\uparrow}^d Z_{1\downarrow}^L & n_{a\uparrow}^c d_{a\uparrow}^\dagger Z_{1\downarrow}^L \\ (1 - n_{a\uparrow}^c)d_{a\uparrow}^\dagger Z_{1\downarrow}^L & n_{a\uparrow}^d c_{a\uparrow} Z_{1\downarrow}^L & (1 - n_{a\uparrow}^c)n_{a\uparrow}^d & c_{a\uparrow}d_{a\uparrow}^\dagger \\ -c_{a\uparrow}^\dagger(1 - n_{a\uparrow}^d)Z_{1\downarrow}^L & n_{a\uparrow}^c d_{a\uparrow} Z_{1\downarrow}^L & d_{a\uparrow}c_{a\uparrow}^\dagger & n_{a\uparrow}^c(1 - n_{a\uparrow}^d) \end{pmatrix}, \quad (6.4)$$

for  $\mathbb{E} = (E^{ij})_{i,j=1,\dots,4} \in gl(2|2)$ .

For  $\mathbb{E} = (E^{ij})_{i,j=1,2,3} \in gl(2|1)$ , it is given by the same matrix without line and row 3

$$\mathbb{E}_{a\uparrow} \equiv \begin{pmatrix} (1 - n_{a\uparrow}^c)(1 - n_{a\uparrow}^d) & -c_{a\uparrow}d_{a\uparrow} & -c_{a\uparrow}(1 - n_{a\uparrow}^d)Z_{1\downarrow}^L \\ c_{a\uparrow}^\dagger d_{a\uparrow}^\dagger & n_{a\uparrow}^c n_{a\uparrow}^d & n_{a\uparrow}^c d_{a\uparrow}^\dagger Z_{1\downarrow}^L \\ -c_{a\uparrow}^\dagger(1 - n_{a\uparrow}^d)Z_{1\downarrow}^L & n_{a\uparrow}^c d_{a\uparrow} Z_{1\downarrow}^L & n_{a\uparrow}^c(1 - n_{a\uparrow}^d) \end{pmatrix}, \quad (6.5)$$

where  $n_{x\alpha}^b = b_{x\alpha}^\dagger b_{x\alpha}$  is the particle number operator for  $b = c, d$ , and  $\alpha = \uparrow, \downarrow$ . We have also standard relations between the operators:

$$\{c_{\alpha x}^\dagger, c_{\beta y}\} = \delta_{xy} \delta_{\alpha\beta}; \quad \{d_{\alpha x}^\dagger, d_{\beta y}\} = \delta_{xy} \delta_{\alpha\beta}; \quad \{c_{\alpha x}, d_{\beta y}\} = 0, \quad \alpha, \beta = \uparrow, \downarrow. \quad (6.6)$$

Note that the operators  $d_\downarrow^\dagger, d_\downarrow$  are not present in the construction, so that one can drop the arrow on the operators  $d_\uparrow^\dagger, d_\uparrow$ .

One can remark that the different choices of mappings  $(i, j)$  on  $\{\bar{\mu}, \bar{\nu}\}$  is equivalent to some transformations on fermionic operators' level (e.g.  $c^\dagger \rightarrow c$ , etc...), therefore all the Hamiltonians are equivalent in this sense and differs one from another by changing the representation (e.g. from electrons to holes).

## 6.2 $gl(2|2) \oplus gl(2)$ model

The Hamiltonian of the model is given by

$$\begin{aligned} H_{gl(2|2) \oplus gl(2)} &= H_{\text{Hub}} + \sum_{x=1}^L (c_{\uparrow x+1}^\dagger c_{\uparrow x} + c_{\uparrow x}^\dagger c_{\uparrow x+1}) (n_{x\uparrow}^d n_{x+1\uparrow}^d - n_{x\uparrow}^d - n_{x+1\uparrow}^d) \\ &+ \sum_{x=1}^L (c_{\uparrow x+1}^\dagger c_{\uparrow x} d_{\uparrow x+1}^\dagger d_{\uparrow x} + c_{\uparrow x}^\dagger c_{\uparrow x+1} d_{\uparrow x}^\dagger d_{\uparrow x+1}) \\ &- 2u \sum_{x=1}^L (1 - 2n_{x\downarrow}^c)(1 - n_{x\uparrow}^c) n_{x\uparrow}^d \\ &+ \sum_{x=1}^L (d_{\uparrow x+1}^\dagger d_{\uparrow x} + d_{\uparrow x}^\dagger d_{\uparrow x+1}) (n_{x\uparrow}^c n_{x+1\uparrow}^c - n_{x\uparrow}^c - n_{x+1\uparrow}^c) \end{aligned} \quad (6.7)$$

with

$$H_{\text{Hub}} = \sum_{\substack{x=1 \\ \alpha=\uparrow, \downarrow}}^L [c_{\alpha x+1}^\dagger c_{\alpha x} + c_{\alpha x}^\dagger c_{\alpha x+1}] + u \sum_{x=1}^L (1 - 2n_{x\downarrow}^c)(1 - 2n_{x\uparrow}^c). \quad (6.8)$$

The eigenfunctions for this Hamiltonian are made of creator operators  $c_{\uparrow}^{\dagger}, d_{\uparrow}^{\dagger}$  and  $c_{\downarrow}^{\dagger}$ . They can be written in the following form and correspond to the solutions found in the previous sections:

$$\begin{aligned} \Phi_{N_1, N_2, N_3}^{(n)} &= \sum_{\vec{z}} \sum_{\vec{y}} \sum_{\vec{x}} \Psi'(\vec{x}, \vec{y}, \vec{z}) \prod_{j=1}^{N_1-n} c_{x_j \uparrow}^{\dagger} \prod_{j=N_1+1}^{N_1+N_2-n} d_{x_j \uparrow}^{\dagger} \prod_{k=1}^n c_{y_k \uparrow}^{\dagger} d_{y_k \uparrow}^{\dagger} \prod_{j=1}^{N_3} c_{z_j \downarrow}^{\dagger} |0\rangle \\ &= (-1)^{N_1+N_3} \phi[\bar{A}], \quad \text{with } n = 0, \dots, \min(N_1, N_2) \end{aligned} \quad (6.9)$$

$$\text{and } \bar{A} = \left( \overbrace{4 \uparrow, \dots, 4 \uparrow}^{N_1-n}, \overbrace{3 \uparrow, \dots, 3 \uparrow}^{N_2-n}, \overbrace{2 \uparrow, \dots, 2 \uparrow}^n, \overbrace{2 \downarrow, \dots, 2 \downarrow}^{N_3} \right). \quad (6.10)$$

$\phi[\bar{A}]$  is the eigenfunction given in (4.10). Remark that, in addition to the particles  $c_{\uparrow}^{\dagger}, d_{\uparrow}^{\dagger}$  and  $c_{\downarrow}^{\dagger}$  (corresponding to  $4 \uparrow, 3 \uparrow$  and  $2 \downarrow$  resp.), we have a doublet  $c_{\uparrow}^{\dagger} d_{\uparrow}^{\dagger}$  corresponding to  $2 \uparrow$ . The particles  $c_{\uparrow}^{\dagger}$  and  $c_{\downarrow}^{\dagger}$  can be identified with a (spin up and down) electron, while  $d_{\uparrow}^{\dagger}$  can be viewed as a spin 0 fermion that can form bound state with the spin up electron.

The energy of the excited state  $\Phi_{N_1, N_2, N_3}^{(n)}$  reads

$$E^{(n)} = L - 2(N_1 + N_2 + N_3 - n) + 2 \sum_{l=1}^{N_1+N_2+N_3-n} \cos k_l,$$

where the parameters  $k_l$  are Bethe roots defined by equations given in section 4.1 (with  $\mathbf{n} = \mathbf{m} = 2$ ).

### 6.3 $gl(2|1) \oplus gl(2)$ model

The Hamiltonian of the model is given by

$$\begin{aligned} H_{gl(2|1) \oplus gl(2)} &= H_{\text{Hub}} + \sum_{x=1}^L (c_{\uparrow x+1}^{\dagger} c_{\uparrow x} + c_{\uparrow x}^{\dagger} c_{\uparrow x+1}) (n_{\uparrow x}^d n_{x+1}^d - n_{\uparrow x}^d - n_{\uparrow x+1}^d) \\ &\quad + \sum_{x=1}^L \left( c_{\uparrow x+1}^{\dagger} c_{\uparrow x} d_{\uparrow x+1}^{\dagger} d_{\uparrow x} + c_{\uparrow x}^{\dagger} c_{\uparrow x+1} d_{\uparrow x}^{\dagger} d_{\uparrow x+1} \right) \\ &\quad - u \sum_{x=1}^L (1 - 2n_{x \downarrow}^c) (1 - n_{x \uparrow}^c) n_{\uparrow x}^d \end{aligned} \quad (6.11)$$

where  $H_{\text{Hub}}$  has been given in (6.8). Again, the eigenfunctions for this Hamiltonian correspond to the solutions found in the previous sections. They have the form (for  $N_2 \leq N_1$ )

$$\begin{aligned} \Phi_{N_1, N_2, N_3} &= \sum_{\vec{z}} \sum_{\vec{y}} \sum_{\vec{x}} \Psi(\vec{x}, \vec{y}, \vec{z}) \prod_{l=1}^{N_1} c_{x_l \uparrow}^{\dagger} \prod_{l=1}^{N_2} c_{y_l \uparrow}^{\dagger} d_{y_l \uparrow}^{\dagger} \prod_{l=1}^{N_3} c_{z_l \downarrow}^{\dagger} |0\rangle \\ &= (-1)^{N_1+N_3} \phi[\bar{A}], \quad \text{with } \bar{A} = \left( \overbrace{3 \uparrow, \dots, 3 \uparrow}^{N_1}, \overbrace{2 \uparrow, \dots, 2 \uparrow}^{N_2}, \overbrace{2 \downarrow, \dots, 2 \downarrow}^{N_3} \right) \end{aligned} \quad (6.12)$$

$\phi[\bar{A}]$  has been defined in (3.14) and the corresponding eigenvalue reads

$$E = L - 2(N_1 + N_2 + N_3) + 2 \sum_{l=1}^{N_1+N_2+N_3} \cos k_l$$

with Bethe roots  $k_l$  obeying the equations given in section 3.2.

## 7 Conclusion

In this paper we presented the Bethe equations using the coordinate Bethe ansatz for  $gl(2|1) \oplus gl(2)$  and  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(2)$  generalized Hubbard model. We wrote explicitly the Hamiltonians for several cases in terms of fermionic creation and annihilation operators. Clearly, the full derivation of the Bethe Ansatz Equations for generalized Hubbard models [16, 17] has to be accomplished: the case of  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(3)$  and its generalizations to  $gl(\mathfrak{n}|\mathfrak{m}) \oplus gl(\mathfrak{n}'|\mathfrak{m}')$  algebras are presently under investigation.

Applications to condensed matter physics deserve also some attention. The models presented in section 6 are generalization of Hubbard models to several types of fermions. They could be of some relevance to systems where electrons of different ‘types’ or ‘colors’ occur (for instance on some ladder spin chain).

Although the link with AdS/CFT correspondence is not direct, the present construction gives a way to introduce a phase in the Bethe equations of Hubbard type models (see discussion in section 5.1). It is thus worthwhile to look deeper at these models. They could be good candidates for an integrable model close to the one underlying the SYM theory, or give a new point of view for the wrapping problem.

Obviously, in all cases, the thermodynamical limit of these models needs also to be investigated.

## References

- [1] J. Hubbard, *Electron Correlations in Narrow Energy Bands*, *Proc. Roy. Soc. Lond. A* **276** (1963) 238; *Electron Correlations in Narrow Energy Bands. II. The Degenerate Band Case*, *Proc. Roy. Soc. Lond. A* **277** (1964) 237.
- [2] M.C. Gutzwiller, *Effect of Correlation on the Ferromagnetism of Transition Metals*, *Phys. Rev. Lett.* **10** (1963) 15.
- [3] N.F. Mott, *Metal-Insulator Transition*, *Rev. Mod. Phys.* **40** (1968) 677 [SPIRES].
- [4] J. Hubbard, *Electron Correlations in Narrow Energy Bands. III. An Improved Solution*, *Proc. Roy. Soc. Lond. A* **281** (1964) 401.
- [5] P.W. Anderson, *The Resonating Valence Bond State in  $La_2CuO_4$  and Superconductivity*, *Science* **235** (1987) 1196.
- [6] I. Affleck and J.B. Marston, *Large- $n$  limit of the Heisenberg-Hubbard model: implications for high- $T_c$  superconductors*, *Phys. Rev. B* **37** (1987) 3774 [SPIRES].
- [7] E.H. Lieb and F.Y. Wu, *Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension*, *Phys. Rev. Lett.* **20** (1968) 1445 [Erratum *ibid.* **21** (1968) 192] [SPIRES]; *The one-dimensional Hubbard model: a reminiscence*, *Physica A* **321** (2003) 1 [cond-mat/0207529].
- [8] F. Eßler, H. Frahm, F. Goehmann, A. Klumper and V. Korepin, *The One-Dimensional Hubbard Model*, Cambridge University Press, Cambridge, U.K. (2005).
- [9] B.S. Shastry, *Infinite conservation laws in the one-dimensional Hubbard model*, *Phys. Rev. Lett.* **56** (1986) 1529 [Erratum *ibid.* **56** (1986) 2334]; *Exact Integrability of the One-Dimensional Hubbard Model*, *Phys. Rev. Lett.* **56** (1986) 2453 [SPIRES]; *Decorated star-triangle relations and exact integrability of the one-dimensional Hubbard model*, *J. Stat. Phys.* **50** (1988) 57.

- [10] E. Olmedilla, M. Wadati and Y. Akutsu, *Yang-Baxter Relations for Spin Models and Fermion Models*, *J. Phys. Soc. Jpn.* **56** (1987) 2298.
- [11] M. Shiroishi and M. Wadati, *Yang-Baxter equation for the R-matrix of the one-dimensional Hubbard model*, *J. Phys. Soc. Jpn.* **64** (1995) 57.
- [12] F.H.L. Eßler, V.E. Korepin and K. Schoutens, *Completeness of the  $SO(4)$  extended Bethe ansatz for the one-dimensional Hubbard model*, *Nucl. Phys. B* **384** (1992) 431 [[cond-mat/9209012](#)] [[SPIRES](#)].
- [13] F.C. Alcaraz and R.Z. Bariev, *Interpolation between Hubbard and supersymmetric  $t$ - $J$  models: two-parameter integrable models of correlated electrons*, *J. Phys. A* **32** (1999) L483 [[cond-mat/9908265](#)].
- [14] X.-W. Guan et al., *Integrable variant of the one-dimensional Hubbard model*, *J. Math. Phys.* **43** (2002) 3445 [[cond-mat/0103059](#)].
- [15] Z. Maassarani and P. Mathieu, *The  $su(N)$   $XX$  model*, *Nucl. Phys. B* **517** (1998) 395 [[cond-mat/9709163](#)];  
Z. Maassarani, *The  $su(N)$  Hubbard model*, *Phys. Lett. A* **239** (1998) 187 [[cond-mat/9709252](#)]; *Exact integrability of the  $su(n)$  Hubbard model*, *Mod. Phys. Lett. B* **12** (1998) 51 [[cond-mat/9710083](#)].
- [16] J.M. Drummond, G. Feverati, L. Frappat and É. Ragoucy, *Super-Hubbard models and applications*, *JHEP* **05** (2007) 008 [[hep-th/0703078](#)] [[SPIRES](#)].
- [17] G. Feverati, L. Frappat and É. Ragoucy, *Universal Hubbard models with arbitrary symmetry*, *J. Stat. Mech.* (2009) P04014 [[arXiv:0903.0190](#)] [[SPIRES](#)].
- [18] A. Rej, D. Serban and M. Staudacher, *Planar  $N = 4$  gauge theory and the Hubbard model*, *JHEP* **03** (2006) 018 [[hep-th/0512077](#)] [[SPIRES](#)].
- [19] B. Sutherland, *An introduction to the Bethe ansatz*, *Lect. Notes Phys.* **242** (1985) 1, B. Shastry, S. Jha and V. Singh eds., Springer, Berlin, Germany.
- [20] A. Rej, M. Staudacher and S. Zieme, *Nesting and dressing*, *J. Stat. Mech.* (2007) P08006 [[hep-th/0702151](#)] [[SPIRES](#)].
- [21] P. Jordan and E.P. Wigner, *Über das Paulische Äquivalenzverbot*, *Zeit. Phys.* **47** (1928) 631 [[SPIRES](#)].